Perturbed linear rough differential equations*

Laure Coutin[†] Antoine Lejay[‡]

August 6, 2012

Abstract

We study linear rough differential equations and we solve perturbed linear rough differential equation with a representation using the Duhamel principle. These results provide us with the key technical point to study the regularity of the differential of the Itô map in a subsequent article.

1 Introduction

Linear Rough Differential Equations (RDE) have been considered by several authors since they are an essential tool for studying the derivative of the Itô map and its flow properties (See the bibliography in [15] for the many references about the flow properties). However, at the exception of the works of D. Feyel, A. de la Pradelle and G. Mokobodzki [21] and S. Aida [1], linear RDE have hardly been considered as objects as such with their own properties, except to control the growth of the solution as in [23, 31]. Instead, they are generally presented as a special case of RDE. However, the Baker-Campbell-Hausdorff-Dynkin formula was among the inspirations of the theory of rough paths [2, 7]. Indeed, the algebraic view of solution of controlled differential equations through for example Chen-Fliess series as well as some numerical simulation algorithms stem directly from the theory of linear controlled ordinary differential equations (See [6] for an overview and [23, 25, 38, 39] for the relationship

*This work has been supported by the ANR Ecru (ANR-09-BLAN-0114-01/02). A. Lejay also want to thank the Centre IntraFacultaire Bernoulli (CIB) and the NSF (National Science Foundation) for its kind hospitality during the *SPDE Semester* in 2012.

†Institut de Mathématiques de Toulouse, F-31062 Toulouse Cedex 9, France. E-mail: `laure.coutin@math.univ-toulouse.fr`

‡Université de Lorraine, IECN, UMR 7502, Vandœuvre-lès-Nancy, F-54600, France
CNRS, IECN, UMR 7502, Vandœuvre-lès-Nancy, F-54600, France
Inria, Villers-lès-Nancy, F-54600, France
E-mail: `Antoine.Lejay@iecn.u-nancy.fr`

between algebra and RDE). Linear equations are among the first examples given in the article [38] and the book [39] as motivation for developing rough paths theory.

The main goal of this article is then to study linear RDE in a general setting by extending some of the results of [21]. Our initial motivation was to extend the Duhamel principle to deal with sensitivity analysis of the Itô map.

The content of this article may be applied to bounded linear operators on an infinite dimensional Banach space. Of course, dealing with unbounded family of operators, for example for solving Stochastic Partial Differential Equations, is much more intricate and needs specific treatments. The reader is referred to the quickly growing literature on these subjects [8,9,25–27]... The Variation of constant principle is also linked to Volterra equations which have been studied in the rough path context by A. Deya [17,18].

In Section 2, we define for any $p \geq 1$ the notion of *rough resolvent* which is a family of linear operators giving the solutions to the linear RDE. We study its main properties and give a representation in term of Dyson like series (or Neumann expansion) for equations of type

$$y_t = y_0 + \sum_{i=1}^{d} \int_0^t y_s B^i \,\mathrm{d}x_s^i$$

for a family of matrices $B^i \in \mathcal{M}_{d\times d}(\mathbb{R})$ and a path x of finite p-variation extendeded to a rough path by completing some of the results in [38, Section 2.4.1]. Besides, we establish that the Chen-Strichartz [42] or Magnus formula [6] holds for any geometric rough path and for any (s,t) such that $\omega(s,t)$ is small enough. This means that the resolvent $\mathsf{A}_{s,t}$ which is defined so that $y_t = y_0\mathsf{A}_{s,t}$ may be written as

$$\mathsf{A}_{s,t} = \exp\left(\sum_{k\geq 1} \sum_{\text{words } I \text{ of length } k} \beta^I \mathbf{x}_{s,t}^I\right)$$

where $\sum_{\text{words } I \text{ of length } k} \mathbf{x}^I$ is the component of the full rough path $\mathbf{x}$ in $(\mathbb{R}^d)^{\otimes k}$ and β^I is a linear combinations of the iterated Lie brackets $[\cdots[B^{i_1}, B^{i_2}], \cdot, B^{i_k}]$ for $i_1, \dots, i_k \in \{1, \dots, d\}$. Here, the exponential is the matrix exponential and the series in the exponential is normally convergent in (s,t). This result then extended to the rough path situation the results of K.-T. Chen [13,14] and of R. Strichartz [42]. This kind of expansion has already been studied for the Brownian motion [5,11,12,35] and the fractional Brownian motion [4] and may serves as basis for numerical simulations [12,35]. However, it was not clear from the current literature that the involved series converges in general excepted for differentiable paths x with bounded derivatives (See [6,40] or the discussion in [42] and in [4] for the fractional Brownian motion).

In Section 3, we specialize our results for linear RDEs of type

$$y_t = y_0 + \int_0^t \mathrm{d}\mathcal{A}_s y_s, \tag{1}$$

where $\mathcal{A}$ is a path of finite p-variation $p \in [1,2)$ with values in the set of linear operators. We give a variation of constant principle for the solution to

$$y_t = y_0 + \int_0^t \mathrm{d}\mathcal{A}_s y_s + (b_t - b_0) \tag{2}$$

and a representation in terms of Dyson series:

$$\mathsf{B}_t = \left(\mathsf{Id} + \sum_{k\geq 1} \mathsf{A}_t^{(k)}\right) y_0 \text{ with } \mathsf{A}_t^{(1)} = \mathcal{A}_t - \mathcal{A}_0 \text{ and } \mathsf{A}_t^{(k+1)} := \int_0^t \mathrm{d}\mathcal{A}_s \mathsf{A}_{0,s}^{(k)},$$

as well as the Magnus formula in short time.

In Section 4, we consider the case of linear RDE for $\mathcal{A}$ of finite p-variation with $p \in [2,3)$. For operator-valued paths of finite p-variation with $p \in [1,3)$, we provide a Duhamel principle/Variation of constant principle for a perturbed linear equation. In particular, it is required to extend properly the notion of perturbation and define a proper notion of integral, and then to lift $\mathcal{A}$ as well as the perturbation b to consider linear RDE of type (1) or perturbed linear RDE of type (2).

In Section 4.3, we show that our notion is well adapted for the kind of linear equations which arises when one differentiates the Itô map with respect to its starting point. In a subsequent article [15], we use these properties to show that the Itô map is differentiable with a Lipschitz or Hölder continuous Fréchet derivative.

2 Rough resolvent

2.1 Almost rough resolvent and rough resolvent

From now, C denotes a constant whose value may vary from line to line.

Let $\Delta_2(T) := \{(s,t) | 0 \leq s \leq t \leq T\}$ and $\Delta_3(T) := \{(s,r,t) | 0 \leq s \leq r \leq t \leq T\}$. Let $\omega : \Delta_2(T) \to \mathbb{R}_+$ be a control, that is a function which is continuous close to its diagonal and such that $\omega(s,r) + \omega(r,t) \leq \omega(s,t)$ for all $(s,r,t) \in \Delta_3(T)$.

Let $\mathfrak{L}$ be a unital algebra over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ which is a Banach space with a norm $\|\cdot\|$ such that $\|AB\| \leq \|A\| \cdot \|B\|$ for any $A, B \in \mathfrak{L}$, $\|\lambda A\| = |\lambda| \cdot \|A\|$ for $\lambda \in \mathbb{K}$ and $\|\mathsf{Id}\| = 1$, where Id is the neutral (left and right) element for the multiplication. This space is called a *Banach algebra* [19].

Remark 1. For $k = 1, \dots, \infty$, let $\mathrm{T}_k(\mathbb{R}^d)$ be the tensor algebra defined by $\mathrm{T}_k(\mathbb{R}^d) := \mathbb{R} \oplus \mathbb{R}^d \oplus (\mathbb{R}^d)^{\otimes 2} \oplus \cdots \oplus (\mathbb{R}^d)^{\otimes k}$ with the addition $+$ and the tensor product $\otimes$. The tensor space $(\mathbb{R}^d)^{\otimes \ell}$ is endowed with a norm $|\cdot|$ such that $|a \otimes b| \leq |a| \cdot |b|$ for any $a \in (\mathbb{R}^d)^{\ell'}$ and $b \in (\mathbb{R}^d)^{\ell - \ell'}$, $\ell' = 1, \dots, \ell - 1$. For such a choice of norm which transforms $(\mathbb{R}^d)^{\otimes \ell}$ into Banach algebra, see [19, Ex. 1.36 and 2.31]. The norm we use on $\mathrm{T}_k(\mathbb{R}^d)$ is then

$$\|x\| = \sum_{\ell=0}^{k} |x_\ell| \text{ when } x = \sum_{\ell=0}^{k} x_\ell \text{ with } x_\ell \in (\mathbb{R}^d)^{\otimes \ell}.$$

The tensor space $\mathrm{T}_k(\mathbb{R}^d)$ corresponds to the space $\mathrm{T}_\infty(\mathbb{R}^d)$ quotiented by the relation $x \sim y$ when $x - y \in (\mathbb{R}^d)^{\otimes(k+1)} \oplus (\mathbb{R}^d)^{(\otimes(k+2)} \oplus \cdots$. For any $k = 1, \dots, \infty$, $\mathrm{T}_k(\mathbb{R}^d)$ becomes a Banach algebra when equipped with the norm $\|\cdot\|$ and the unit $1 \in \mathbb{R}$.

For $A \in \mathfrak{L}$ such that $\|A - \mathsf{Id}\| < 1$,

$$A^{-1} := \sum_{i=0}^{+\infty} (-1)^k (A - \mathsf{Id})^k.$$

is a left and right inverse of A.

Typically, we will consider for $\mathfrak{L}$ the space $L(\mathrm{V}, \mathrm{V})$ of linear bounded operators on a Banach space V with a norm $|\cdot|$, which we equip with the norm $\|A\| = \sup_{u \in \mathrm{V},\ |u|=1} |Au|$.

Definition 1 (Rough resolvent)**.** A family $\{\mathsf{A}_{s,t}\}_{(s,t)\in\Delta_2(T)}$ of operators in $\mathfrak{L}$ satisfying

$$\begin{aligned} \mathsf{A}_{s,t} = \mathsf{A}_{r,t}\mathsf{A}_{s,r} \text{ (resp. } \mathsf{A}_{s,t} = \mathsf{A}_{s,r}\mathsf{A}_{r,t}),\ \forall (s,r,t) \in \Delta_3(T), \\ \|\mathsf{A}_{s,t} - \mathsf{Id}\| \leq C\omega(s,t)^{1/p},\ \forall (s,t) \in \Delta_2(T) \end{aligned} \tag{3}$$

for some constant C is called a *p-left rough resolvent* (resp. *p-right rough resolvent*)

Our main tool is the following. An alternative and elegant proof may be found in [21, Theorem 10] for some $\omega(s,t) = V(t-s)$.

Following [10], we introduce

$$\tau_{i+1} = \inf\left\{ t > \tau_i,\ (1 + B\omega(\tau_i, t)^{1/p})^2 + C4^\theta \omega(\tau_i, t)^\theta \sum_{r=1}^{+\infty} \frac{1}{r^\theta} \leq 2^{\theta/2} \right\} \text{ with } \tau_0 = 0,$$
$$N_{T,\omega} = \sup\{n \in \mathbb{N},\ \tau_n < T\}.$$

Theorem 1. *Let $(\mathsf{B}_{s,t})_{(s,t)\in\Delta_2(T)}$ be a family of operators in $\mathfrak{L}$ such that for some $\theta > 1$ and some constants C and B,*

$$\|\mathsf{B}_{s,t} - \mathsf{Id}\| \leq B\omega(s,t)^{1/p},\ \forall (s,t) \in \Delta_2(T) \tag{4}$$

$$\|\mathsf{B}_{s,t} - \mathsf{B}_{r,t}\mathsf{B}_{s,r}\| \leq C\omega(s,t)^\theta,\ \forall (s,r,t) \in \Delta_3(T). \tag{5}$$

Then there exists a unique left p-rough resolvent $\{\mathsf{A}_{s,t}\}_{(s,t)\in\Delta_2(T)}$ such that

$$\|\mathsf{A}_{s,t} - \mathsf{B}_{s,t}\| \leq 2^{\frac{\theta}{2}(N_{T,\omega}+1)} LC(2 + B\omega(0,T)^{1/p})\omega(s,t)^\theta, \tag{6}$$

for all $(s,t) \in \Delta_2(T)$ *where* $L = 4^\theta \sum_{r=1}^\infty r^{-\theta}$*. Besides,* $\mathsf{A}_{s,t}$ *is invertible for any* $(s,t) \in \Delta_2(T)$*. Such a family is then a* p-left rough resolvent*.*

Similarly, a p*-right rough resolvent satisfying* (6) *may be associated to a family of operators* $\{\mathsf{B}_{s,t}\}_{(s,t)\in\Delta_2(T)}$ *in* $\mathfrak{L}$ *satisfying* (4) *and*

$$\|\mathsf{B}_{s,t} - \mathsf{B}_{s,r}\mathsf{B}_{r,t}\| \le C\omega(s,t)^\theta \text{ for } (s,r,t) \in \Delta_3(T).$$

Definition 2. A family $\{\mathsf{B}_{s,t}\}_{(s,t)\in\Delta_2(T)}$ of operators satisfying the above hypotheses is called an *almost left-resolvent*.

Remark 2. The family of operators B and A necessarily satisfies $\mathsf{B}_{t,t} = \mathsf{Id} = \mathsf{A}_{t,t}$ for any $t \in [0,T]$.

Remark 3. If $\mathsf{B}_{s,t}$ takes its values in a Lie group $\mathfrak{G}$ (See [2] for an introduction) for any $(s,t) \in \Delta_2(T)$, then $\mathsf{A}_{s,t}$ also takes its values in $\mathfrak{G}$ for any $(s,t) \in \Delta_2(T)$.

Remark 4. For $\mathfrak{L} = \mathrm{T}_k(\mathbb{R}^d)$, then one recovers the sewing lemma which allows one to transform an almost rough path to a rough path. When $\mathfrak{L} = \mathrm{T}_\infty(\mathbb{R}^d)$ and $\mathsf{B}_{s,t} = \mathbf{x}_{s,t}$ for a rough path $\mathbf{x}$ with values in $\mathrm{T}_{\lfloor p\rfloor}(\mathbb{R}^d)$ of finite p-variation controlled by ω, then Theorem 1 gives the extension of $\mathbf{x}$ to $\mathrm{T}_\infty(\mathbb{R}^d)$, which the the Chen series associated to $\mathbf{x}$ [13].

Proof of Theorem 1. Let us prove first uniqueness of a p-rough resolvent satisfying (6). Note first that any p-rough resolvent which satisfies (6) also satisfies an inequality of type (4) with a constant C'. Then, for $(s,t) \in \Delta_2(T)$ such that $C'\omega(s,t)^{1/p} < 1$, $\mathsf{A}_{s,t}$ is invertible. Using the property $\mathsf{A}_{s,t} = \mathsf{A}_{r,t}\mathsf{A}_{s,t}$, it is then easily obtained that $\mathsf{A}_{s,t}$ is invertible for any $(s,t) \in \Delta_2(T)$.

Now, let A and C be two left p-rough resolvent satisfying (6) and set $f(t) = \mathsf{C}_{0,t}^{-1}\mathsf{A}_{0,t}$. Hence,

$$f(t) - f(s) = \mathsf{C}_{0,t}^{-1}[\mathsf{A}_{s,t} - \mathsf{C}_{s,t}]\mathsf{A}_{0,s}.$$

Note that $\|\mathsf{C}_{s,t}^{-1}\| \le \frac{1}{1-C\omega(s,t)}$, for $C\omega(s,t) < 1$. Using property $\mathsf{C}_{0,t} = \mathsf{C}_{s,t}\mathsf{C}_{0,s}$, we obtain $\sup_{t\in[0,T]} \|\mathsf{C}_{0,t}^{-1}\| < +\infty$. With (6), one gets that $\|\mathsf{A}_{s,t} - \mathsf{C}_{s,t}\| \le C\omega(s,t)^\theta$ for some constant C. Then f is of finite θ-variation with $\theta > 1$. This means that f is constant and equal to $f(0) = \mathsf{Id}$. This proves the uniqueness of a p-rough resolvent satisfying (6).

Let us now prove the existence of a p-rough resolvent. Our proof consists in mixing the proof of Theorem 3.2.1 in [39] and the one of Theorem 7 in [21].

Fix $(\sigma,\tau) \in \Delta_2(T)$ and let $D = \{\sigma = t_0 < t_1 < \cdots < t_r = \tau\}$ be a partition of $[\sigma,\tau]$. Set

$$\mathsf{A}(D)_{s,t} = \mathsf{B}_{t_j,t}\mathsf{B}_{t_{j-1},t_j}\cdots\mathsf{B}_{t_i,t_{i+1}}\mathsf{B}_{s,t_i}$$

where $t_{i-1} < s \le t_i$ and $t_j \le t < t_{j+1}$ with by convention $t_{-1} = -\infty$ and $t_{r+1} = +\infty$. If not such interval $[t_i,t_j]$ exists, then set $\mathsf{A}_{s,t}(D) = \mathsf{Id}$. When $r \ge 3$, it is possible to choose t_ℓ such that

$$\omega(t_{\ell-1}, t_{\ell+1}) \leq \frac{2\omega(\sigma,\tau)}{r-1}. \tag{7}$$

With such a choice t_ℓ, set $D' = D \setminus \{t_\ell\}$. Then if $t_\ell \in (t_i, t_j)$,

$$\mathsf{A}(D)_{s,t} - \mathsf{A}(D')_{s,t} = \mathsf{B}_{t_j,t}\left(\prod_{m=0}^{j-\ell-2} \mathsf{B}_{t_{j-1-m},t_{j-m}}\right)\left(\mathsf{B}_{t_\ell,t_{\ell+1}}\mathsf{B}_{t_{\ell-1},t_\ell} - \mathsf{B}_{t_{\ell-1},t_\ell}\right) \cdot \left(\prod_{m=0}^{\ell-i-2} \mathsf{B}_{t_{\ell-2-m},t_{\ell-1-m}}\right)\mathsf{B}_{s,t_i},$$

if $t_\ell = t_j$,

$$\mathsf{A}(D)_{s,t} - \mathsf{A}(D')_{s,t} = \left(\mathsf{B}_{t_\ell,t}\mathsf{B}_{t_{\ell-1},t_\ell} - \mathsf{B}_{t_{\ell-1},t}\right) \cdot \left(\prod_{m=0}^{\ell-i-2} \mathsf{B}_{t_{\ell-2-m},t_{\ell-1-m}}\right)\mathsf{B}_{s,t_i},$$

if $t_\ell = t_i$,

$$\mathsf{A}(D)_{s,t} - \mathsf{A}(D')_{s,t} = \mathsf{B}_{t_j,t}\left(\prod_{m=0}^{j-\ell-2} \mathsf{B}_{t_{j-1-m},t_{j-m}}\right)\left(\mathsf{B}_{t_\ell,t_{\ell+1}}\mathsf{B}_{s,t_\ell} - \mathsf{B}_{s,t_\ell}\right).$$

Otherwise, $\mathsf{A}(D)_{s,t} = \mathsf{A}(D')_{s,t}$. Let

$$h_r(\sigma,\tau) = \sup_{\substack{D \text{ partition of } [\sigma,\tau],\\ |D|\leq r}} \sup_{\substack{(s,t)\in\Delta_2(T)\\ [s,t]\subset[\sigma,\tau]}} \|\mathsf{A}(D)_{s,t}\|, \text{ for } r \geq 2,$$

$$U_r(\sigma,\tau) = \sup_{\substack{D \text{ partition of } [\sigma,\tau],\\ |D|\leq r}} \sup_{\substack{(s,t)\in\Delta_2(T)\\ [s,t]\subset[\sigma,\tau]}} \|\mathsf{A}(D)_{s,t} - \mathsf{A}(D')_{s,t}\|, \text{ for } r \geq 3.$$

With (5), we deduce that for $r \geq 2$,

$$h_{r+1}(\sigma,\tau) \leq h_r(\sigma,\tau) + U_{r+1}(\sigma,\tau),$$
$$U_{r+1}(\sigma,\tau) \leq h_r(\sigma,\tau)^2 \frac{2^\theta C\omega(\sigma,\tau)^\theta}{(r-1)^\theta},$$

where C is the constant appearing in (5). With (4) and (5),

$$h_2(\sigma,\tau) \leq (1 + B\omega(\sigma,\tau)^{1/p})^2.$$

Let us choose σ and τ so that

$$(1 + B\omega(\sigma,\tau)^{1/p})^2 + C4^\theta\omega(\sigma,\tau)^\theta \sum_{r=1}^{+\infty} \frac{1}{r^\theta} \leq 2^{\theta/2}, \tag{8}$$

where B is the constant appearing in (4). This choice is possible since ω is continuous close to its diagonal and $\theta > 1$.

This implies that $h_2(\sigma, \tau) \leq 2^{\theta/2}$ and that

$$U_3(\sigma, \tau) \leq 4^\theta C \frac{\omega(\sigma, \tau)^\theta}{(3-1)^\theta} \text{ and } h_3(\sigma, \tau) \leq h_2(\sigma, \tau) + U_3(\sigma, \tau) \leq 2^{\theta/2}.$$

We now assume that for $3 \leq i \leq r$,

$$U_i(\sigma, \tau) \leq 4^\theta C \frac{\omega(\sigma, \tau)^\theta}{(i-2)^\theta} \text{ and } h_i(\sigma, \tau) \leq 2^{\theta/2}. \tag{9}$$

With the induction hypothesis (9),

$$U_{r+1}(\sigma, \tau) \leq 4^\theta \frac{C\omega(\sigma, \tau)^\theta}{(r-1)^\theta}.$$

Besides, with (8),

$$\begin{aligned} h_{r+1}(\sigma, \tau) &\leq h_2(\sigma, \tau) + U_{r+1}(\sigma, \tau) + U_r(\sigma, \tau) + \cdots + U_3(\sigma, \tau) \\ &\leq (1 + B\omega(\sigma, \tau)^{1/p})^2 + 4^\theta C \sum_{i=2}^{r} \frac{\omega(\sigma, \tau)^\theta}{(i-1)^\theta} \leq 2^{\theta/2}. \end{aligned}$$

This proves that (9) holds for any $r \geq 3$.

Hence, for any partition D of $[\sigma, \tau]$ with (σ, τ) satisfying (8), this proves that

$$\|\mathsf{A}(D)_{s,t} - \mathsf{B}_{s,t}\| \leq LC\omega(\sigma, \tau)^\theta \tag{10}$$

with $L = 4^\theta \sum_{i=1}^\infty \frac{1}{i^\theta}$. For this, it is sufficient to remove successively some points of D in a way which satisfies (7).

One the other hand, choose τ_1 such that $(0, \tau_1)$ satisfies (8). Construct a sequence of partitions D^n with $D^1 = \{0, \tau_1\}$ and by adding successively a point an in interval $[t_{\ell-1}, t_{\ell+1}]$ of D^n such that (7) is satisfied and the mesh of the partition decreases to 0 as $n \to \infty$. Our previous controls (9) on $U_r(0, \tau_1)$ and $h_r(0, \tau_1)$ shows that $\{\mathsf{A}(D^n)_{s,t}\}_{n \geq 2}$ is a Cauchy sequence, with a limit $\mathsf{A}_{s,t}$ for any $(s, t) \in \Delta_2(\tau_1)$ uniformly.

Using a diagonal extraction argument, it is possible to assert that $\mathsf{A}(D^n)_{s,t}$ converges to $\mathsf{A}_{s,t}$ for any $(s, t) \in \Delta_2(\tau_1) \cap \mathbb{Q}^2$.

Let us remark that $\mathsf{A}(D^n)_{s,t} = \mathsf{A}(D^n_{s,t})_{s,t}$ with $D^n_{s,t} = (D^n \cap [s, t]) \cup \{s, t\}$. Passing to the limit in (10), this proves that A satisfies $\|\mathsf{A}_{s,t} - \mathsf{B}_{s,t}\| \leq LC\omega(s, t)^\theta$ for $(s, t) \in \Delta_2(\tau_1) \cap \mathbb{Q}^2$. Hence, we may extend $\mathsf{A}_{s,t}$ to $\Delta_2(\tau_1)$ by continuity since $(s, t) \mapsto \mathsf{B}_{s,t}$ is continuous on $\Delta_2(\tau_1)$, as well as $\omega(s, t)$.

Moreover, for $0 \leq s \leq r \leq t \leq \tau_1$, using (18),

$$\|\mathsf{A}(D^n)_{s,t} - \mathsf{A}(D^n)_{r,t}\mathsf{A}(D^n)_{s,r}\| \leq 2^\theta C\omega(t_{i-1}, t_{i+1})^\theta$$

where $t_i \in D^n$ is such that $t_i \leq r < t_{i+1}$. Passing to the limit, $\mathsf{A}_{s,t} = \mathsf{A}_{r,t}\mathsf{A}_{s,r}$ for $(s,r,t) \in \Delta_3(\tau_1)$ since ω is continuous close to its diagonal. By continuity, this is true for any $(s,r,t) \in \Delta_3(\tau_1)$.

Since ω is continuous close to its diagonal and $[0,T]$ is compact, $N_{T,\omega} < +\infty$. We extend A to $\Delta_2(T)$ by defining first A on an interval $[\tau_i \wedge T, \tau_{i+1} \wedge T]$, $i \leq N_{T,\omega}$. For s in $[\tau_i, \tau_{i+1}]$ and $t \in [\tau_j, \tau_{j+1}]$ with $i < j$, we set $\mathsf{A}_{s,t} = \mathsf{A}_{\tau_j,t}\mathsf{A}_{\tau_{j-1},\tau_j} \cdots \mathsf{A}_{\tau_{i+1},\tau_{i+2}}\mathsf{A}_{s,\tau_{i+1}}$.

With (8) and our definition of τ_i,

$$\|\mathsf{B}_{s,t}\| \leq 2^{\theta/4} \text{ and } \|\mathsf{A}_{s,t}\| \leq 2^{\theta/2} \text{ for } (s,t) \in [\tau_i, \tau_{i+1} \wedge T],\ i = 0, \dots, N_{T,\omega}.$$

For $(s,r,t) \in \Delta_3(T)$,

$$\mathsf{A}_{s,t} - \mathsf{B}_{s,t} = \mathsf{A}_{r,t}(\mathsf{A}_{s,r} - \mathsf{B}_{s,r}) + (\mathsf{A}_{r,t} - \mathsf{B}_{r,t})\mathsf{B}_{s,r} + \mathsf{B}_{r,t}\mathsf{B}_{s,r} - \mathsf{B}_{s,t}$$

Set

$$u_i := \sup_{(s,t)\in\Delta(\tau_{i+1})} \frac{\|\mathsf{A}_{s,t} - \mathsf{B}_{s,t}\|}{\omega(s,t)^\theta}$$

and note that $u_0 \leq LC$ from (19). For $t \in [\tau_i, \tau_{i+1}]$ and $s < \tau_i$ and $r = \tau_i$, then

$$\|\mathsf{A}_{s,t} - \mathsf{B}_{s,t}\| \leq 2^{\theta/2} u_{i-1}\omega(s,r)^\theta + MLC\omega(r,t)^\theta + C\omega(s,t)^\theta$$

with $M = \sup_{(s,t)\in\Delta_2(T)} \|\mathsf{B}_{s,t}\| \leq 1 + B\omega(0,T)^{1/p}$. With the super-additivity of ω,

$$u_i \leq \max\{2^{\theta/2}u_{i-1}, MLC\} + C \leq 2^{\theta/2}u_{i-1} + C(ML+1).$$

Hence,

$$\begin{aligned}\sup_{(s,t)\in\Delta_2(T)} \|\mathsf{A}_{s,t} - \mathsf{B}_{s,t}\| &\leq \frac{2^{(N_{T,\omega}+1)\theta/2} - 1}{2^{\theta/2} - 1} \max\{CL, C(ML+1)\} \\ &\leq 2^{(N_{T,\omega}+1)\theta/2} CL(M+1)\end{aligned}$$

since $L > 1$.

It is easily checked that $\{\mathsf{A}_{s,t}\}_{(s,t)\in\Delta_2(T)}$ satisfies all the required properties. □

Corollary 1. *Let $\{\mathsf{B}_{s,t}\}_{(s,t)\in\Delta_2(T)}$ satisfying the hypotheses of Theorem 1 and let $\{\mathsf{C}_{s,t}\}_{(s,t)\in\Delta_2(T)}$ be a family of operators satisfying for some $\theta > 1$, $\|\mathsf{C}_{s,t} - \mathsf{B}_{s,t}\| \leq C\omega(s,t)^\theta$ for all $(s,t) \in \Delta_2(T)$. Then C is an almost resolvent and the associated resolvent is the one generated by B.*

Proof. It is easily seen that C satisfies (4)-(5). The proof follows then from the uniqueness in Theorem 1. □

The proof of the next continuity result is similar to the one of Theorem 1 so we skip it. It is the linear counterpart of Theorem 3.2.2 in [39].

Corollary 2 (Continuity property)**.** *Let* B *and* B' *be two almost right* p*-rough resolvents such that for some* $\epsilon \geq 0$*,*

$$\|\mathsf{B}_{s,t} - \mathsf{B}'_{s,t}\| \leq \epsilon\omega(s,t)^{1/p} \text{ and } \|\mathsf{B}_{s,t} - \mathsf{B}'_{s,t} - \mathsf{B}_{s,r}\mathsf{B}_{r,t} + \mathsf{B}'_{s,r}\mathsf{B}'_{r,t}\| \leq \epsilon\omega(s,t)^{\theta}$$

for any $(s,r,t) \in \Delta_3(T)$*. Then there exists a constant* L *such that*

$$\|\mathsf{A}_{s,t} - \mathsf{B}_{s,t} - \mathsf{A}'_{s,t} - \mathsf{B}'_{s,t}\| \leq \epsilon L\omega(s,t)^{\theta},$$

where A *(resp* A'*) is the* p*-rough resolvent associated to* B *(resp.* B'*).*

A similar result holds for almost left p*-rough resolvent.*

Let $(\mathrm{V}, |\cdot|)$ be a Banach space and V^* be its dual.

A linear operator in $L(\mathrm{V},\mathrm{V})$ will be seen as an operator acting on the left, while an operator an operator in $L(\mathrm{V}^*,\mathrm{V}^*)$ be will seen as an operator acting on the right. These spaces are endowed with the norm operator, denoted by $\|\cdot\|$. When V is a finite dimensional space, an element of V will be identified to a column vector, while an element of V^* will be identified to a row vector. A linear application in $L(\mathrm{V},\mathrm{V})$ or $L(\mathrm{V}^*,\mathrm{V}^*)$ will then be identified with a matrix.

For $p \geq 1$, let $\mathcal{R}^p(\mathrm{V})$ be the set of paths with values in V such that $|x_0| < +\infty$ and

$$\|x\|_p := \sup_{(s,t)\in\Delta_2(T)} \frac{|x_{s,t}|}{\omega(s,t)^{1/p}} < +\infty \text{ with } x_{s,t} := x_t - x_s.$$

An element in $\mathcal{R}^p(\mathrm{V})$ is called a function of *finite* p*-variation controlled by* ω. Under the condition that $\omega(s,t) = t - s$, then paths in $\mathcal{R}^p$ are α-Hölder continuous with $\alpha = 1/p$.

Corollary 3. *Let* B *be an almost left (resp. right)* p*-rough resolvent with values in* $L(\mathrm{V},\mathrm{V})$ *and* A *be its associated left (resp. right)* p*-rough resolvent as in Theorem 1. Let* $z \in \mathcal{R}^p(\mathrm{V})$ *be such that*

$$|z_t - \mathsf{B}_{s,t}z_s| \leq C\omega(s,t)^{\theta} \text{ (resp. } |z_s - \mathsf{B}_{s,t}z_t| \leq C\omega(s,t)^{\theta}\text{)}, \ \forall (s,t) \in \Delta_2(T).$$

Then $z_t = \mathsf{A}_{0,t}z_0$ *(resp.* $z_0 = \mathsf{A}_{0,t}z_t$*),* $t \in [0,T]$*.*

Proof. Since $\mathsf{A}_{0,t} = \mathsf{A}_{s,t}\mathsf{A}_{0,s}$ is invertible and $\mathsf{A}_{0,t}^{-1} = \mathsf{A}_{0,s}^{-1}\mathsf{A}_{s,t}^{-1}$, set $u_t := \mathsf{A}_{0,t}^{-1}z_t$ so that

$$u_t = \mathsf{A}_{0,t}^{-1}(z_t - \mathsf{B}_{s,t}z_s) + \mathsf{A}_{0,t}^{-1}(\mathsf{B}_{s,t} - \mathsf{A}_{s,t})z_s + u_s.$$

This proves that u is of finite θ-variation with $\theta > 1$ and is then constant and equal to z_0.

For right p-rough resolvent, set $u_t = \mathsf{A}_{0,t}^{-1}z_0$ and the same result holds. □

2.2 Constructing right p-rough resolvents from a left p-rough resolvent

Let us consider a p-rough left resolvent $\{\mathsf{A}_{s,t}\}_{(s,t)\in\Delta_2(T)}$ in $\mathfrak{L}$. Then there are at least two ways to construct a p-rough right resolvent.

Proposition 1. *If $\{\mathsf{A}_{s,t}\}_{(s,t)\in\Delta_2(T)}$ is a left p-rough resolvent in $\mathfrak{L}$ then $\{\mathsf{A}_{T-t,T-s}\}_{(s,t)\in\Delta_2(T)}$ is a right p-rough resolvent in $\mathfrak{L}$*

Proof. The proof is immediate. □

Proposition 2. *If $\{\mathsf{A}_{s,t}\}_{(s,t)\in\Delta_2(T)}$ in $\mathfrak{L}$ is a left p-rough resolvent, then $\mathsf{A}_{s,t}$ is invertible for any $(s,t)\in\Delta_2(T)$ and $\{\mathsf{A}_{s,t}^{-1}\}_{(s,t)\in\Delta_2(T)}$ is a p-right rough resolvent in $\mathfrak{L}$.*

Proof. This follows from Theorem 1. □

2.3 An alternative version

As said above, the proof of Theorem 1 relies on mixing two approaches. We could have also used the neo-classical inequality (See [39, Theorem 3.1.1]) to get the following control. As the proof is similar to the one of Theorem 3.1.2 of [39] (See also Theorem 4.1 in [1] for the linear case), we skip it. See also [36] for a continuity result. This approach is also used by P. Friz and N. Victoir in [23] for weak geometric rough paths. When $\omega(s,t)=t-s$, D. Feyel *et al.* [21] obtain the same result by noting that $\lambda\mapsto\sum_{k=0}^{\lfloor p\rfloor}\lambda^k\mathsf{B}_{s,t}^{(k)}$ for $\lambda\in\mathbb{C}$ and $\mathsf{B}^{(k)}$ satisfying (11)-(12) below is an analytic map and an almost resolvent. The map is then transformed using their multiplicative sewing lemma into an analytic map $\lambda\mapsto\sum_{k=+\infty}^{+\infty}\lambda^k\mathsf{B}_{s,t}^{(k)}$ for $(s,t)\in\Delta_2(T)$ which is a resolvent.

Let $\Gamma(z)=\int_0^{+\infty}e^{-t}t^{z-1}\,\mathrm{d}t$ be the Gamma function.

Theorem 2. *Let $p\geq 1$ and $(\mathsf{B}_{s,t}^{(k)})_{(s,t)\in\Delta_2(T)}$, $k=0,\dots,\lfloor p\rfloor$, be a family of operators in $\mathfrak{L}$ such that*

$$\mathsf{B}^{(0)}=\mathsf{Id}, \tag{11}$$

$$\mathsf{B}_{s,t}^{(k)}=\sum_{i=0}^{k}\mathsf{B}_{r,t}^{(i)}\mathsf{B}_{s,r}^{(k-i)},\ (s,r,t)\in\Delta_3(T),\ 0\leq k\leq\lfloor p\rfloor,$$

$$\|\mathsf{B}_{s,t}^{(k)}\|\leq\frac{\omega(s,t)^{k/p}}{\beta\Gamma(k/p)}, \tag{12}$$

where β is a constant such that $\beta\geq p^2\left(1+\sum_{r=1}^{\infty}\left(\frac{2}{r}\right)^{\frac{\lfloor p\rfloor+1}{p}}\right)$. Then, $\mathsf{B}=\sum_{i=0}^{[p]}\mathsf{B}^{(k)}$ is an almost left p-rough resolvent. Moreover, there exists a unique sequence $(\mathsf{A}^{(k)})_{(s,t)\in\Delta_2(T)}$,

$k \in \mathbb{N}$ *of operators in* $\mathfrak{L}$ *such that Inequality* (12) *holds for* $k \in \mathbb{N}$, $\mathsf{A}^{(k)} = \mathsf{B}^{(k)}$, $0 \leq k \leq \lfloor p \rfloor$ *and* $\mathsf{A} = \sum_{k=0}^{\infty} \mathsf{A}^{(k)}$ *is the left* p*-rough resolvent associated to* B. *Of course, a similar result holds for right resolvents.*

The following control follows from the inequality shown at the end of the proof of Theorem 4.1 in [1] (See also the proof of Corollary 5 below).

Corollary 4. *Let* A *be a* p*-rough resolvent constructed from an almost resolvent* B *as above. Then there exist some constants* C_1 *and* C_2 *depending only on* β *and* p *such that*

$$\|\mathsf{A}_{s,t}\| \leq C_1(1 + \omega(s,t)) \exp(C_2 \omega(s,t)), \forall (s,t) \in \Delta_2(T).$$

2.4 Series representation of the resolvent

In this section and the following, we consider the case of controlled linear RDE, that is, of RDE corresponding formally to equations of type

$$y_t = y_0 + \int_0^t \sum_{i=1}^d y_s \mathcal{B}^i \,\mathrm{d}x_s^i,$$

where $\mathcal{B}^1, \dots, \mathcal{B}^d$ are some matrices in $\mathcal{M}_{m\times m}(\mathbb{R})$ or $\mathcal{M}_{m\times m}(\mathbb{C})$, and x^i are the components of the projection of a rough path of finite p-variation $\mathbf{x}$. If x is smooth, then

$$y_t = y_0 + y_0 \sum_{i=1}^d \mathcal{B}^i x_{0,t}^i + \int_0^t \int_0^s \sum_{i,j=1}^d y_r \mathcal{B}^j \mathcal{B}^i \,\mathrm{d}x_r^j \,\mathrm{d}x_s^i, \tag{13}$$

and so on...

Following the formalism developed by K.T. Chen [14] and the one of Magnus formula [6], which was one of the inspiration of rough paths theory (See [38]), our aim is then to construct first a representation of the resolvent by a series of operators and then as the exponential of another series.

Let $\mathrm{T}_{\lfloor p \rfloor}(\mathbb{R}^d)$ be the tensor space as defined in Remark 1.

Let $\mathbf{x}$ be a rough path of finite p-variation, $p \geq 1$ with values in $\mathrm{T}_{\lfloor p \rfloor}(\mathbb{R}^d)$. This means that $\mathbf{x}$ is a path from $[0,T]$ to $\mathrm{T}_{\lfloor p \rfloor}(\mathbb{R}^d)$ whose component in $\mathbb{R}$ is 1 so that $\mathbf{x}_t$ is invertible. The increment of $\mathbf{x}$ is $\mathbf{x}_{s,t} := \mathbf{x}_s^{-1} \otimes \mathbf{x}_t$ and $\mathbf{x}_t = \mathbf{x}_{0,t}$. For any $(s,t) \in \Delta_2(T)$, the components of $\mathbf{x}_{s,t}$ in $(\mathbb{R}^d)^k$ may be decomposed as $\mathbf{x}_{s,t}^{(k)} = \sum_{\text{words } I \text{ s.t. } |I| = k} \mathbf{x}_{s,t}^I$, where a word is a concatenation of letters in $\{1, \dots, d\}$, and the component of $\mathbf{x}_{s,t}$ in $\mathbb{R}$, corresponding to the empty word $\emptyset$, is $\mathbf{x}_{s,t}^{\emptyset} = 1$. In addition,

$$\mathbf{x}_{s,t} = \mathbf{x}_{s,r} \otimes \mathbf{x}_{r,t} \text{ for any } (s,r,t) \in \Delta_3(T), \tag{14}$$

$$\sum_{\text{words } I \text{ s.t. } |I| = k} |\mathbf{x}_{s,t}^I| \leq C_k \omega(s,t)^{k/p} \text{ for } (s,t) \in \Delta_2(T),\ k = 1, \dots, \lfloor p \rfloor. \tag{15}$$

The set of such rough paths is denoted by $\mathcal{R}^p(\mathrm{T}_{\lfloor p\rfloor}(\mathbb{R}^d))$. We refer to [23, 32, 34, 38, 39] for the properties of a rough paths.

Let us assume that C_k in (15) is such that

$$C_k \leq \frac{1}{\beta\Gamma(k/p)},\ k = 1,\dots,\lfloor p\rfloor, \tag{16}$$

where β is as in Theorem 2. This is always possible up to changing ω.

From the extension theorem [39, Theorem 3.1.2], for any $m \geq \lfloor p\rfloor$, $\mathbf{x}$ may be uniquely extended to a rough path with values in $\mathrm{T}_m(\mathbb{R}^d)$ in a way such that (14) and (15) are satisfied.

Let $\mathcal{B}^1, \cdots, \mathcal{B}^d$ a family of matrices in $\mathcal{M}_{d'\times d'}(\mathbb{K})$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. For a word $I = i_1 \cdots i_k$, we set $\mathcal{B}^I = \mathcal{B}^{i_1} \cdots \mathcal{B}^{i_k}$.

With $\widetilde{\omega}(s,t) := \max_{i=1,\dots,d} \|\mathcal{B}^i\|^p \omega(s,t)$, it holds that

$$\sum_{\text{words } I \text{ s.t. } |\mathrm{I}|=\mathrm{k}} \|\mathcal{B}^I \mathbf{x}^I_{s,t}\| \leq \frac{\widetilde{\omega}(s,t)^{k/p}}{\beta\Gamma(k/p)} \text{ for } k = 1,\dots,\lfloor p\rfloor. \tag{17}$$

Using the unique extension of $\mathbf{x}$ to a rough path with values $\mathrm{T}_m(\mathbb{R}^d)$ for $m \geq \lfloor p\rfloor$, let us set $\mathsf{B}^{(0)} = \mathsf{Id}$ and

$$\mathsf{B}^{(k)}_{s,t} := \sum_{\text{words } I \text{ s.t. } |\mathrm{I}|=\mathrm{k}} \mathcal{B}^I \mathbf{x}^I_{s,t} \text{ for } 1 \leq k \leq m.$$

Equation (13) justifies to consider $\mathsf{A}^{(k)} := \sum_{\ell=0}^k \mathsf{B}^{(\ell)}$ as an almost rough resolvent.

Corollary 5. *For $k \geq \lfloor p\rfloor$, $\mathsf{A}^{(k)} := \sum_{\ell=0}^k \mathsf{B}^{(\ell)}$ is an almost right p-rough resolvent whose associated right p-rough resolvent is equal to*

$$\mathsf{A}_{s,t} = \sum_{k=0}^{+\infty} \sum_{\textit{words } I \textit{ s.t. } |I| = k} \mathsf{B}^I \mathbf{x}^I_{s,t} \tag{18}$$

and for some universal constant L and any $k \geq \lfloor p\rfloor$,

$$\|\mathsf{A}_{s,t} - \mathsf{A}^{(k)}_{s,t}\| \leq \frac{L(\lfloor p\rfloor + 1)}{\beta\Gamma((k+1)/p)}(1 + \widetilde{\omega}(s,t)) \exp(1 + \widetilde{\omega}(s,t)) \widetilde{\omega}(s,t)^{(k+1)/p}. \tag{19}$$

Besides, the map $\mathbf{x} \in \mathcal{R}^p(\mathrm{T}_{\lfloor p\rfloor}(\mathbb{R}^d)) \mapsto \mathsf{A}$ is continuous.

Remark 5. In Section 3.4.3, this representation will be generalized to matrix valued paths of finite p-variation, $1 \leq p < 2$.

Remark 6. To consider a left linear RDE instead of a right linear RDE, it is sufficient to use for $\mathsf{B}^{(k)}_{s,t}$ the value $\mathsf{B}^{(k)}_{s,t} := \sum_{|I|=k} \mathcal{B}^{\overline{I}} \mathbf{x}^I_{s,t}$ with $\overline{I} = i_k \cdots i_1$ when $I = i_1 \dots i_k$ [13].

Proof. For $k \geq \lfloor p \rfloor$,

$$\mathsf{A}^{(k)}_{s,t} = \mathsf{A}^{(k)}_{s,r}\mathsf{A}^{(k)}_{r,t} + \sum_{\ell=k+1}^{2k} \sum_{\text{words } I, J \text{ s.t. } |IJ|=\ell} \mathcal{B}^{IJ}\mathbf{x}^{I}_{s,r}\mathbf{x}^{J}_{r,t}. \tag{20}$$

Hence, $\mathsf{A}^{(k)}$ is an almost right p-rough resolvent which may be uniquely extended by Theorem 2 to some right p-rough resolvent $\mathsf{A}^{[k]}$. With Corollary 1 applied to $\mathsf{A}^{(k)}$ and $\mathsf{A}^{(\ell)}$ with $k, \ell \geq \lfloor p \rfloor$, $\mathsf{A}^{[k]}$ does not depend on k and so we denote it by A.

Using the neo-classical inequality (See [39, Theorem 3.1.1]) on (20), this proves that

$$\|\mathsf{A}^{(k)}_{s,t} - \mathsf{A}^{(k)}_{s,r}\mathsf{A}^{(k)}_{r,t}\| \leq \sum_{\ell \geq k+1} \frac{\widetilde{\omega}(s,t)^{\ell/p}}{\beta\Gamma(\ell/p)}.$$

As in [1, Theorem 4.2] and using the properties of the Γ function, we have that for $k \geq \lfloor p \rfloor$,

$$\sum_{\ell \geq k+1} \frac{\widetilde{\omega}(s,t)^{(\ell-k-1)/p}}{\Gamma(\ell/p)} \leq \sum_{\ell \geq 0} \frac{(1+\widetilde{\omega}(s,t))^{1+\lfloor \ell/p \rfloor}}{\Gamma((\ell+k+1)/p)} \leq (\lfloor p \rfloor + 1) \sum_{\ell \geq 0} \frac{(1+\widetilde{\omega}(s,t))^{\ell+1}}{\Gamma(\ell + (k+1)/p)},$$

since there are at most $(\lfloor p \rfloor + 1)$ integer values of ℓ such that $\lfloor \ell/p \rfloor = n$ for any integer n, and Γ is increasing. But

$$\Gamma(\ell+\alpha) = \frac{\Gamma(\ell)\Gamma(\alpha)}{\int_0^1 t^{\ell-1}(1-t)^{\alpha-1}\,\mathrm{d}t} \geq \Gamma(\ell)\Gamma(\alpha)$$

for $\ell \geq 1$, $\alpha \geq 1$. Hence,

$$\begin{aligned}\sum_{\ell \geq k+1} \frac{\widetilde{\omega}(s,t)^{(\ell-k-1)/p}}{\Gamma(\ell/p)} &\leq (\lfloor p \rfloor + 1)(1+\widetilde{\omega}(s,t)) \left(\frac{1}{\Gamma((k+1)/p)} + \sum_{\ell \geq 1} \frac{(1+\widetilde{\omega}(s,t))^{\ell}}{\Gamma(\ell)\Gamma((k+1)/p)} \right) \\ &\leq \frac{(\lfloor p \rfloor + 1)(1+\widetilde{\omega}(s,t))}{\Gamma((k+1)/p)} \exp(1+\widetilde{\omega}(s,t)).\end{aligned}$$

Hence, (19) holds for some universal constant L. Letting k going to the infinity yields that A is the resolvent associated to $\mathsf{A}^{(k)}$. This proves that $\mathsf{A}^{(k)}_{s,t}$ converges to A and then that (18) holds.

The continuity of $\mathbf{x} \in \mathcal{R}^p(\mathrm{T}_{\lfloor p \rfloor}(\mathbb{R}^d)) \mapsto \mathsf{A}$ follows from the results in [36]. □

2.5 Exponential representation of the resolvent

We are still in the same framework as above by adding some restrictions the the rough path and on the operators.

We assume that $\mathbf{x}$ is a *weak geometric rough path*, which means that it takes its value in $\mathrm{G}_{\lfloor p\rfloor}(\mathbb{R}^d)$, the free nilpotent Lie algebra of step $\lfloor p\rfloor$ defined the following way: Let $\mathrm{A}_{\lfloor p\rfloor}(\mathbb{R}^d)$ be the smallest subset of $\mathrm{T}_{\lfloor p\rfloor}(\mathbb{R}^d)$ containing $\mathbb{R}^d$ and which is stable under $[\cdot,\cdot]$ defined by $[a,b]=a\otimes b-b\otimes a$ for $a,b\in\mathrm{T}_{\lfloor p\rfloor}(\mathbb{R}^d)$. Let log and exp be the logarithm and exponential in $\mathrm{T}_{\lfloor p\rfloor}(\mathbb{R}^d)$ formally defined by

$$\log(1+a)=\sum_{k=1}^{\lfloor p\rfloor}\frac{(-1)^{k-1}}{k}a^{\otimes k}\text{ and }\exp(a)=1+\sum_{k=1}^{\lfloor p\rfloor}\frac{1}{k!}a^{\otimes k}.$$

for an element $a=a^1+\ldots+a^{\lfloor p\rfloor}$, then $\mathrm{G}_{\lfloor p\rfloor}(\mathbb{R}^d)=\exp(\mathrm{A}(\mathbb{R}^d))$.

A *smooth rough path* is a rough path constructed from a smooth path $x:[0,T]\to\mathbb{R}^d$ by setting

$$\mathbf{x}_{s,t}^{i_1\cdots i_k}=\int_s^t\int_s^{t_1}\cdots\int_s^{t_{k-1}}\dot{x}_{t_1}^{i_1}\cdots\dot{x}_{t_k}^{i_k}\,\mathrm{d}t_1\cdots\mathrm{d}t_k. \tag{21}$$

By definition, a *geometric rough path* is the limit of smooth rough paths with respect to the norm induced by p-variation [24, 39]. Any geometric p-rough path is a weak geometric p-rough path, while any weak geometric p-rough path is a q-rough path for any $q>p$.

Geometric rough paths satisfy the same properties as the truncated Chen series. In particular, $\log(\mathbf{x}_{s,t})\in\mathrm{A}_{\lfloor p\rfloor}(\mathbb{R}^d)$ may be extended as a series involving the terms of $\mathbf{x}_{s,t}$ as in [13, 41, 42] (See also [3, 11, 22, 43] for example for the Brownian case). Then one obtains the following result which is an adaptation of the ones given in these references.

In particular, weak geometric rough paths satisfy the *shuffle property* [13, 41]: for two words I and J,

$$\mathbf{x}^I\mathbf{x}^J=\sum_{K\in\mathrm{Shuffle}(I,J)}\mathbf{x}^K, \tag{22}$$

where $\mathrm{Shuffle}(I,J)$ is the set of words made from the words $I=(i_1,\ldots,i_{|I|})$ and $J=(j_1,\ldots,j_{|J|})$ of type $(k_1,\ldots,k_{|I|+|J|})$ such that each letter of K corresponds exactly to one letter of I or J and the order of the letters of I and the order of the letters in J is preserved.

Lemma 1. *Let $\mathbf{x}$ be a weak geometric rough path satisfying* (15) *and* (16). *Then there exists a constant $\tau>0$ such that for $(s,t)\in\Delta_2(T)$ with $\omega(s,t)<\tau$,*

$$\mathsf{A}_{s,t}:=\exp\left(\sum_{k=1}^{+\infty}\sum_{|I|=k}\alpha^I\mathbf{x}_{s,t}^I\right), \tag{23}$$

where the series in the exponential is normally convergent and each α^I is a linear combination of elements of $\{\mathcal{B}^J\}_{|J|=I}$, independent from $\mathbf{x}$.

Remark 7. For $2 \leq p < 3$ any p-rough path $\mathbf{x}$ with values in $\mathrm{T}_{\lfloor p \rfloor}(\mathbb{R}^d)$ may be decomposed as the sum of a weak geometric p-rough path $\mathbf{y}$ with values in $\mathrm{G}_{\lfloor p \rfloor}(\mathbb{R}^d)$ and a path $\varphi : [0,T] \to \mathcal{M}_{d\times d}(\mathbb{R}^d)$ of finite $p/2$-variation such that $\varphi_t^{i,j} = \varphi_t^{j,i}$ for any $t \in [0,T]$ and all $i,j = 1,\dots,d$ [33]. The almost p-rough resolvent associated by $\mathbf{x}$ and $\{\mathcal{B}^i\}_{i=1\dots,d}$ as given in Corollary 5 is

$$\mathsf{A}_{s,t}^{(2)} = \mathsf{Id} + \sum_{i=1}^{d} \mathcal{B}^i \mathbf{y}_{s,t}^i + \sum_{i,j=1}^{d} \mathcal{B}^i \mathcal{B}^j \mathbf{y}_{s,t}^{i\cdot j} + \sum_{i,j=1}^{d} \mathcal{B}^i \mathcal{B}^j \varphi_{s,t}^{i\cdot j}$$

Since $2 \leq p < 3$, $p/2 < 2$ and $2/p + 1/p > 1$. Being defined as Young integrals, the iterated integrals between $\mathbf{y}$ and φ exists, as well as the iterated integrals between φ and itself. Hence, the path $\mathbf{z}$ defined by

$$\mathbf{z}_t = \mathbf{y}_t + \varphi_t \sum_{i,j=1}^{d} \left(\int_0^t (\varphi_s^i - \varphi_0^i) \otimes \mathrm{d}\varphi_s^j + \int_0^t (\mathbf{y}_s^i - \mathbf{y}_0^i) \otimes \mathrm{d}\varphi_s^j + \int_0^t (\varphi_s^i - \varphi_0^i) \otimes \mathrm{d}\mathbf{y}_s^j \right)$$

is a p-rough path with values in $\mathrm{T}_{\lfloor p \rfloor}(\mathbb{R}^d \oplus \mathbb{R}^d \otimes \mathbb{R}^d)$.

Thanks to Corollary 1, the almost rough resolvent associated to $\mathbf{x}$ and $\{\mathcal{B}^i\}_{i=1,\dots,d}$ is the same as the one associated to $\mathbf{z}$ and $\{\mathcal{B}^i, \mathcal{B}^{j\cdot k}\}_{i,j,k=1,\dots,d}$. Lemma 1 above and Theorem 3 below are then easily extended for non-geometric rough path of finite p-variation when $2 \leq p < 3$. The situation seems more cumbersome when $p \geq 3$.

Proof. Let us set for $\lambda \in \mathbb{C}$,

$$\mathsf{A}_{s,t}(\lambda) := \sum_{k=0}^{+\infty} \lambda^k \mathsf{B}_{s,t}^{(k)}.$$

With Corollary 5, it holds that (17) holds for any integer $k \geq 1$ and then for any $\tau > 0$,

$$\sum_{k=1}^{+\infty} |\lambda|^k \sup_{0\leq s\leq t\leq s+\tau} |\mathsf{B}_{s,t}^{(k)}| \leq \sum_{k=1}^{+\infty} \lambda^k \frac{\sup_{0\leq s\leq t\leq s+\tau} \widetilde{\omega}(s,t)^{k/p}}{\beta\Gamma(k/p)}.$$

With the same majoration as in Corollary 7, this proves that $\mathsf{A}_{s,t}(\lambda) - \mathsf{Id}$ is normally convergent in (s,t) and that there exists a constant K depending only on p, $\omega(0,T)$, $\max_{i=1,\dots,d} \|\mathcal{B}^i\|$ and p such that

$$\|\mathsf{A}_{s,t}(\lambda) - \mathsf{Id}\| \leq K\lambda\omega(s,t)^{1/p},\ \forall |\lambda| \leq 2.$$

Hence, if (s,t) is such that for some $k' > k > 1$, $K\omega(s,t)^{1/p} < 1/k'$ and $|\lambda| < k$, the series defined by

$$\log(\mathsf{A}_{s,t}(\lambda)) = \sum_{n\geq 1} \frac{(-1)^{n+1}}{n} (\mathsf{A}_{s,t}(\lambda) - \mathsf{Id})^n$$

is normally convergent in (s,t) and λ. Furthermore, $\exp(\log(\mathsf{A}_{s,t}(\lambda))) = \mathsf{A}_{s,t}(\lambda)$, where exp is the matrix exponential [2, Prop. 2.3, p. 48]. By composing power series, for some $k' > k > 1$, for $|\lambda| < k$ and $K\omega(s,t)^{1/p} < 1/k'$,

$$\log(\mathsf{A}_{s,t}(\lambda)) = \sum_{k\geq 1} \lambda^k \mathsf{C}_{s,t}^{(k)} \tag{24}$$

with

$$\mathsf{C}_{s,t}^{(k)} = \sum_{m=0}^{k} \frac{(-1)^{m+1}}{m} \sum_{n_1+\dots+n_m=k} \mathsf{B}_{s,t}^{(n_1)} \cdots \mathsf{B}_{s,t}^{(n_m)}.$$

Using the definition of the $\mathsf{B}^{(k)}$ and the shuffle property (22), this yields (23) by taking $\lambda = 1$. □

Let us consider now that case where the matrices $\mathcal{B}^i$, $i = 1,\dots,d$ belongs to a simply connected Lie algebra $\mathfrak{g} \subset \mathcal{M}_{d'\times d'}(\mathbb{K})$ [2,29].

A striking result from K.-T. Chen [13,14] is that $\log(\mathsf{A}_{s,t})$ may be formally expanded as a series of type $\sum_{k\geq 1}\sum_{|I|=k} \beta^I \mathbf{x}^I$, where the β^I indeed belong to the Lie algebra $\mathfrak{g}$ and is the linear combination of $|I|$ iterated Lie brackets of the $\mathcal{B}^i$'s. As shown by R. Ree in [41], the shuffle property is necessary for this. In the matrix case, a simpler result asserts that locally around Id, exp maps $\mathfrak{g}$ to $\mathfrak{G}$ with log as its inverse [29]. Yet applied to $\mathsf{A}_{s,t}$, this result does not give the decomposition as a series whose coefficients involve $\mathbf{x}^I$. Our aim is then to express $\mathsf{A}_{s,t}$ when $\omega(s,t)$ is small enough as the exponential of a series normally convergent whose terms belong to $\mathfrak{g}$. This is indeed an extension of the Chen-Strichartz series [42] or Magnus series [6].

For $I = i_1,\dots,i_k$,

$$\beta^I := \sum_{\sigma\in\mathfrak{S}_k} \frac{(-1)^{e(\sigma)}}{k^2\binom{k-1}{e(\sigma)}} [[\cdots[\mathcal{B}^{i_{\sigma(1)}}, \mathcal{B}^{i_{\sigma(2)}}],\cdots], \mathcal{B}^{i_{\sigma(k)}}] \tag{25}$$

where $\mathfrak{S}_k$ is the set of permutations of $\{1,\dots,k\}$ and $e(\sigma) = \#\{j \in \{1,\dots,k-1\}; \sigma(j) > \sigma(j+1)\}$. Note that $\beta^I \in \mathfrak{g}$ for any word I.

Let us consider a smooth path x which is extended to a smooth rough path $\mathbf{x}$ using (21). Using the Baker-Campbell-Hausdorff-Dynkin formula [7,29], R.S. Strichartz have shown in [42] that the solution to

$$z'_t = z_t \lambda \sum_{i=1}^{d} \mathcal{B}^i \cdot x_t^i$$

may be written formally

$$z_t = z_s \exp\left(\sum_{k\geq 1} \lambda^k \sum_{|I|=k} \beta^I \mathbf{x}_{s,t}^I\right) \tag{26}$$

for any $\lambda \in \mathbb{C}$. Besides, if $\|\mathcal{B}^i \dot{x}_t^i\| \leq M$ for any $t \in [0, T]$, then $|\mathcal{B}^I \mathbf{x}_{s,t}^I| \leq M^k (t-s)^k / k!$ when $|I| = k$ and then for $|\lambda| M(t-s) \leq 1$, the series in the exponential converges normally [42, p. 337–337].

Theorem 3 (Chen-Strichartz series)**.** *Let* $\mathbf{x}$ *be a weak geometric rough path satisfying* (15) *and* (16)*. Then there exists a constant* $\tau > 0$ *such that for* $(s,t) \in \Delta_2(T)$ *with* $\omega(s,t) < \tau$*,*

$$\mathsf{A}_{s,t} = \exp\left(\sum_{k=1}^{+\infty} \sum_{|I|=k} \beta^I \mathbf{x}_{s,t}^I\right), \tag{27}$$

where β^I *is given by* (25)*.*

Remark 8. Let us assume that the smallest Lie algebra $\mathfrak{g}$ generated by $\mathcal{B}^1, \dots, \mathcal{B}^d$ is nilpotent of step n. This means that for $\mathcal{A}$ and $\mathcal{B}$ in $\mathfrak{g}$, $[\mathcal{A}, \mathcal{B}] := \mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A}$ belongs to $\mathfrak{g}$ and the bracket operator $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is such that $[\cdots[\mathfrak{g}, \mathfrak{g}], \cdots, \mathfrak{g}] = 0$ when more that n brackets are involved. In this case, the exponential is a diffeomorphism between the Lie algebra and its Lie group and (27) holds for any $(s,t) \in \Delta_2(T)$.

In Section 4.2, we give an example of application to the Heisenberg group, whose Lie algebra is nilpotent of step 2.

Remark 9. However, in the general case, one cannot expect that (27) holds for any time $(s,t) \in \Delta_2(T)$, since the existence of the real logarithm of a matrix is not always granted. See [37] and [40] for explicit counter-examples and discussion.

Proof. Let x be a smooth path with $|\mathcal{B}^i \dot{x}_t^i| \leq M$ for $i = 1, \dots, d$ and any $t \in [0, T]$. The path x is lifted as a smooth rough path $\mathbf{x}$

If $\mathsf{C} \in \mathcal{M}_{d\times d}(\mathbb{R})$ is such that $\|\mathsf{C}\| < \log(2)$, then $\log(\exp(\mathsf{C})) = \mathsf{C}$ [29, Theorem 2.7].

Let τ be small enough such that the series in the exponential in (26) and in (24) are normally convergent and have a norm smaller than $\log(2)$ for $|\lambda| < 2$ and (s,t) with $\omega(s,t) \leq \tau$. Then for such a choice of (s,t), whatever the smooth rough path x chosen as above,

$$\sum_{|I|=k} \beta^I \mathbf{x}_{s,t}^I = \sum_{|I|=k} \alpha^I \mathbf{x}_{s,t}^I.$$

To identify the coefficients, let us note that for $I = i_1 \cdots i_k$,

$$\left.\frac{\mathrm{d}^k}{\mathrm{d}^k t} \beta^I \mathbf{x}^I\right|_{t=0} = \beta^I \dot{x}_0^{i_1} \cdots \dot{x}_0^{i_k}$$

and a similar relation holds when β^I is replaced by α^I. From the freedom of choice of the path x, this proves that $\alpha^I = \beta^I$ for any word I. □

The question of the maximum time interval on which the involved series converges is treated for example in [6, 40]. For stochastic processes, see [4] regarding the fractional Brownian motion and [5] for the Brownian motion.

Note that other formulas may be used to represent this logarithm [6] whose formula is known under various names (Chen-Strichartz , Chen-Fliess , Magnus, ...). In any case, the Baker-Campbell-Hausdorff-Dynkin formula is the main tool [7].

A large literature is devoted to the construction of numerical procedures relying on this kind of computation for smooth paths [6, 28], or for Brownian paths (See [12, 35] for example).

3 Linear and linear perturbed equations when $1 \leq p < 2$

3.1 Young integrals

Let us recall some elementary facts on Young integrals [21, 39, 44].

Let X, Y and Z be Banach spaces with a product $(x, y) \in \mathrm{X} \times \mathrm{Y} \mapsto xy \in \mathrm{Z}$.

Let x and y be two paths of finite p-variation controlled by ω with $p < 2$ respectively with values in X and Y. Hence for any $(s, t) \in \Delta_2(T)$, $\int_s^t y_r \,\mathrm{d}x_r$ may be defined as a *Young integral*, that is as limit of Riemann sums $\sum_{i=0}^{n-1} y_{t_i^n}(x_{t_{i+1}^n} - x_{t_i^n})$ for partitions $\{t_i^n\}_{i=0}^n$ whose mesh decrease to 0. Besides,

$$\left|\int_s^t y_r \,\mathrm{d}x_r - y_s(x_t - x_s)\right| \leq \zeta(2p)\|x\|_p\|y\|_p\omega(s, t)^{2/p}. \tag{28}$$

with $\zeta(q) = \sum_{n\geq 1} 1/n^q$, $q > 1$. Note that from (28), $t \mapsto \int_0^t y_r \,\mathrm{d}x_r$ is of finite p-variation controlled by ω and

$$\left\|\int_0^\cdot y_r \,\mathrm{d}x_r\right\|_p \leq (|y_0| + \|y\|_p\omega(0, T)^{1/p})\|x\|_p + \zeta(2p)\|x\|_p\|y\|_p\omega(0, T)^{1/p}. \tag{29}$$

This construction is very general. We will either use $\mathrm{X} = \mathrm{Y} = \mathrm{Z} = \mathfrak{L}$ for an algebra $\mathfrak{L}$ defined as in Section 2.1, or $\mathrm{X} = \mathrm{Z} = \mathrm{V}$ and $\mathrm{Y} = L(\mathrm{V}, \mathrm{V})$ for a Banach space V.

3.2 Linear RDE, $1 \leq p < 2$

In this section we follow and extend the ideas introduced in [21].

Let $\mathcal{A} : [0, T] \to \mathfrak{L}$ be continuous and of finite p-variation controlled by ω and A be defined by $\mathsf{A}_{s,t} := \mathsf{Id} + \mathcal{A}_{s,t}$ with $\mathcal{A}_{s,t} := \mathcal{A}_t - \mathcal{A}_s$ for $(s, t) \in \Delta_2(T)$.

It is easily checked that for all $(s,t) \in \Delta_2(T)$,

$$\|\mathsf{Id} + \mathcal{A}_{s,t} - (\mathsf{Id} + \mathcal{A}_{r,t})(\mathsf{Id} + \mathcal{A}_{s,r})\| = \|\mathcal{A}_{r,t}\mathcal{A}_{s,r}\| \leq C\omega(s,t)^{2p},$$

and then that A is an almost left p-rough resolvent. A similar inequality also holds for $\|\mathsf{A}_{s,t} - \mathsf{A}_{s,r}\mathsf{A}_{r,t}\|$ so that A is also an almost right p-rough resolvent.

Let $\mathsf{L}^{\mathcal{A}}$ and $\mathsf{R}^{\mathcal{A}}$ be the left and right rough resolvent in $\mathfrak{L}$ associated to $\mathsf{A} \in \mathcal{R}^p(\mathfrak{L})$ through Theorem 1.

Lemma 2. *For $\mathcal{A} \in \mathcal{R}^p(\mathfrak{L})$, $\mathsf{L}^{-\mathcal{A}}_{0,t}\mathsf{R}^{\mathcal{A}}_{0,t} = \mathsf{R}^{\mathcal{A}}_{0,t}\mathsf{L}^{-\mathcal{A}}_{0,t} = \mathsf{Id}$ for all $t \in [0,T]$.*

Proof. Note that

$$(\mathcal{A}_{s,t} - \mathsf{Id})(\mathcal{A}_{s,t} + \mathsf{Id}) - \mathsf{Id} = \mathcal{A}_{s,t}\mathcal{A}_{s,t}. \tag{30}$$

Hence

$$\begin{aligned}
\mathsf{R}^{\mathcal{A}}_{0,t} &= \mathsf{R}^{\mathcal{A}}_{0,s}(\mathsf{R}^{\mathcal{A}}_{s,t} - \mathsf{Id} - \mathcal{A}_{s,t}) + \mathsf{R}^{\mathcal{A}}_{0,s}(\mathsf{Id} + \mathcal{A}_{s,t}),\\
\mathsf{L}^{-\mathcal{A}}_{0,t} &= (\mathsf{L}^{-\mathcal{A}}_{s,t} - \mathsf{Id} + \mathcal{A}_{s,t})\mathsf{L}^{-\mathcal{A}}_{0,s} + (\mathsf{Id} - \mathcal{A}_{s,t})\mathsf{L}^{-\mathcal{A}}_{0,s},
\end{aligned}$$

so that for $(s,t) \in \Delta_2(T)$,

$$\mathsf{R}^{\mathcal{A}}_{0,t}\mathsf{L}^{-\mathcal{A}}_{0,t} - \mathsf{R}^{\mathcal{A}}_{0,s}\mathsf{L}^{-\mathcal{A}}_{0,s} = \sigma(s,t) + \mathsf{R}^{\mathcal{A}}_{0,s}\mathcal{A}_{s,t}\mathcal{A}_{s,t}\mathsf{L}^{-\mathcal{A}}_{0,s},$$

where $\|\sigma(s,t)\| \leq C\omega(s,t)^{2p}$ for some constant C, since

$$\|\mathsf{R}^{\mathcal{A}}_{s,t} - \mathsf{Id} - \mathcal{A}_{s,t}\| \leq C'\omega(s,t)^{2p} \text{ and } \|\mathsf{L}^{-\mathcal{A}}_{s,t} - \mathsf{Id} + \mathcal{A}_{s,t}\| \leq C'\omega(s,t)^{2p}$$

and $(s,t) \in \Delta_2(T) \mapsto \|\mathsf{R}^{\mathcal{A}}_{s,t}\| + \|\mathsf{L}^{-\mathcal{A}}_{s,t}\|$ is bounded. Then, $t \mapsto \mathsf{R}^{\mathcal{A}}_{0,t}\mathsf{L}^{-\mathcal{A}}_{0,t}$ is of finite $2p$-variations with values in $\mathfrak{L}$. Since $2p > 1$, this means that $\mathsf{R}^{\mathcal{A}}_{0,t}\mathsf{L}^{-\mathcal{A}}_{0,t}$ is constant. With Remark 2, $\mathsf{R}^{\mathcal{A}}_{0,t}\mathsf{L}^{-\mathcal{A}}_{0,t} = \mathsf{Id}$.

This proves that $\mathsf{R}^{\mathcal{A}}_{0,t}$ is the left inverse of $\mathsf{L}^{-\mathcal{A}}_{0,t}$. Since $\mathsf{R}^{\mathcal{A}}_{0,t}$ and $\mathsf{L}^{-\mathcal{A}}_{0,t}$ are invertible, $\mathsf{R}^{\mathcal{A}}_{0,t}$ and $\mathsf{L}^{-\mathcal{A}}_{0,t}$ are reciprocal inverses. □

Let now V be a Banach space and set $\mathfrak{L} = L(\mathrm{V},\mathrm{V})$.

On the other hand, for $\mathcal{A} \in \mathcal{R}^p(L(\mathrm{V},\mathrm{V}))$ and $u \in \mathcal{R}^p(\mathrm{V})$, one may define the Young integral $\int_0^t \mathrm{d}\mathcal{A}_s u_s$ as a path with values in $\mathcal{R}^p(\mathrm{V})$. By definition, for $(s,t) \in \Delta_2(T)$, for a family of partitions $\{t_i^n\}_{i=0,\dots,n}$ of $[s,t]$ whose mesh decreases to 0,

$$\int_s^t \mathrm{d}\mathcal{A}_r u_r = \lim_{n\to\infty} \sum_{i=0}^{n-1} \mathcal{A}_{t_i^n,t_{i+1}^n} u_{t_i}.$$

Besides, it satisfies for $(s,t) \in \Delta_2(T)$,

$$\left| \int_s^t \mathrm{d}\mathcal{A}_r u_r - \mathcal{A}_{s,t}u_s \right| \leq K\|u\|_p\|\mathcal{A}\|_p\omega(s,t)^{2p}, \tag{31}$$

for some universal constant K.

The next proposition shows the equivalence between several objects we have constructed. A function $u \in \mathcal{R}^p(\mathrm{V})$ satisfying (33) below is said to be a solution in the *sense of Davie* [16].

Proposition 3. *The unique solution in* $\mathcal{R}^p(\mathrm{V})$ *to*

$$u_t = u_0 + \int_0^t \mathrm{d}\mathcal{A}_s u_s \tag{32}$$

which is equal to $u_t = \mathsf{L}^{\mathcal{A}}_{0,t} u_s$, *which is also the unique function* $u \in \mathcal{R}^p(\mathrm{V})$ *satisfying*

$$|u_t - u_s - \mathcal{A}_{s,t} u_s| \leq L\omega(s,t)^{2p} |u_s|, \ \forall (s,t) \in \Delta_2(T). \tag{33}$$

Besides, for any $t \geq 0$, *the application* $a \mapsto \mathsf{L}^{\mathcal{A}}_{0,t} a$ *defines a flow of homeomorphisms, since* $\mathsf{R}^{-\mathcal{A}}_{0,t} \mathsf{L}^{\mathcal{A}}_{0,t} a = a$ *for any* $a \in \mathrm{V}$ *and any* $t \geq 0$.

A similar result holds for right linear equations: $v_t = v_0 + \int_0^t v_s \,\mathrm{d}\mathcal{A}_s$ *with* $v \in \mathcal{R}^p(\mathrm{V}^*)$ *when* $\mathcal{A} \in \mathcal{R}^p(L(\mathrm{V}^*, \mathrm{V}^*))$.

Proof. For $u_t := \mathsf{L}^{\mathcal{A}}_{0,t} u_0$ and $v_t := u_0 + \int_0^t \mathrm{d}\mathcal{A}_r u_r$, with (31) and (6),

$$\begin{aligned} |v_t - v_s - (u_t - u_s)| &\leq |v_t - v_s - \mathcal{A}_{s,t} u_s - (u_t - u_s - \mathcal{A}_{s,t} u_s)| \\ &\leq L\omega(s,t)^{2p} + K\|u\|_p \|\mathcal{A}\|_p \omega(s,t)^{2p}. \end{aligned}$$

Since $2p > 1$, $v - u$ is constant and $v_0 = u_0$, which proves that $v = u$ and that u is solution to (32).

Inequality (33) is a direct consequence of (6). The same argument as above show that a function in $\mathcal{R}^p(\mathrm{V})$ with $u_0 = a$ given is equal to $\mathsf{L}^{\mathcal{A}}_{0,\cdot} a$.

Existence follows from Corollary 3. □

It is now possible to consider operator valued linear RDE. Let Y and Z be some paths with values in $L(\mathrm{V}, \mathrm{V})$ which are solutions to

$$\mathsf{Y}_t = \mathsf{Id} + \int_0^t \mathrm{d}\mathcal{A}_s \mathsf{Y}_s \text{ and } \mathsf{Z}_t = \mathsf{Id} - \int_0^t \mathsf{Z}_s \,\mathrm{d}\mathcal{A}_s.$$

Since the product of two bounded operators in $L(\mathrm{V}, \mathrm{V})$ remains a bounded operator in $L(\mathrm{V}, \mathrm{V})$, the existence and uniqueness of Y and Z is granted by the computations in the proof of Proposition 3. Besides,

$$\mathsf{Y}_t = \mathsf{L}^{\mathcal{A}}_{s,t} \mathsf{Y}_s \text{ and } \mathsf{Z}_t = \mathsf{Z}_s \mathsf{R}^{-\mathcal{A}}_{s,t} \text{ so that } \mathsf{Y}_t = \mathsf{L}^{\mathcal{A}}_{0,t} \text{ and } \mathsf{Z}_t = \mathsf{R}^{-\mathcal{A}}_{0,t}$$

because $\mathsf{Y}_0 = \mathsf{Z}_0 = \mathsf{Id}$. Hence, $\mathsf{Z}_t = \mathsf{Y}_t^{-1}$ for any $t \in [0, T]$.

By linearity, the solution to (32) is equal to $u_t = \mathsf{Y}_t u_0$, $t \in [0, T]$.

3.3 Perturbation of linear differential equation

Definition 3 (Perturbed linear RDE, $1 \leq p < 2$)**.** Let $b^* \in \mathcal{R}^p(\mathrm{V}^*)$ (resp. $b \in \mathcal{R}^p(\mathrm{V})$) and $\mathcal{A} \in \mathcal{R}^p(L(\mathrm{V}^*, \mathrm{V}^*))$ (resp. $\mathcal{A} \in \mathcal{R}^p(L(\mathrm{V}, \mathrm{V}))$). A solution a the perturbed right (resp. left) linear RDE is defined by a function $y \in \mathcal{R}^p(\mathrm{V}^*)$ (resp. $y \in \mathcal{R}^p(\mathrm{V})$) satisfying equivalently for all $(s,t) \in \Delta_2(T)$,

$$|y_t - y_s \mathsf{R}^{\mathcal{A}}_{s,t} - b^*_{s,t}| \leq C\omega(s,t)^\theta \text{ (resp. } |y_t - \mathsf{L}^{\mathcal{A}}_{s,t} y_s - b_{s,t}| \leq C\omega(s,t)^\theta), \tag{34}$$

$$|y_t - y_s - y_s \mathcal{A}_{s,t} - b^*_{s,t}| \leq C\omega(s,t)^\theta \text{ (resp. } |y_t - y_s - \mathcal{A}_{s,t} y_s - b_{s,t}| \leq C\omega(s,t)^\theta), \tag{35}$$

$$y_t = y_0 + \int_0^t y_s \,\mathrm{d}\mathcal{A}_{0,s} + b^*_{0,t}, \text{ (resp.} y_t = y_0 + \int_0^t \mathrm{d}\mathcal{A}_{0,s} y_s + b_{0,t}) \tag{36}$$

with $b^*_{s,t} := b^*_t - b^*_s$ (resp. $b_{s,t} := b_t - b_s$).

If it exists, the uniqueness of y follows from the linearity property. The equivalence between these definitions follows from computations similar to the one of Proposition 3.

Lemma 3. *If $y \in \mathcal{R}^p(\mathrm{V}^*)$ or $y \in \mathcal{R}^p(\mathrm{V})$ satisfies (36), then*

$$\|y\|_p \leq C(|y_0| + \|b\|_p)$$

for a constant C that depends only on p, $\omega(0,T)$ and $\|\mathcal{A}\|_p$.

Proof. It is immediate from the properties of the Young integral that for some constant K,

$$\|y\|_p \leq \|y\|_\infty \|\mathcal{A}\|_p + K\|\mathcal{A}\|_p \|y\|_p \omega(0,T)^{1/p} + \|b\|_p.$$

Since $\|y\|_\infty \leq \|y\|_p \omega(0,T)^{1/p} + |y_0|$, choosing first T small enough so that

$$\omega(0,T)^{1/p} \|\mathcal{A}\|_p (1+K) \leq 1/2$$

allows one to get the desired control first for small time, and then for any time (See [30] for details). □

Let us also note that with (3),

$$\|\mathsf{R}^{\mathcal{A}}_{0,t} - \mathsf{R}^{\mathcal{A}}_{0,s}\| = \|\mathsf{R}^{\mathcal{A}}_{0,s}(\mathsf{Id} - \mathsf{R}^{\mathcal{A}}_{s,t})\| \leq D\omega(s,t)^{1/p} \sup_{r \in [0,T]} \|\mathsf{R}^{\mathcal{A}}_{0,r}\|$$

so that $t \mapsto \mathsf{R}^{\mathcal{A}}_{0,t}$ belongs to $\mathcal{R}^p(L(\mathrm{V}^*, \mathrm{V}^*))$. Similar computations show that $s \in [0,t] \mapsto \mathsf{R}^{\mathcal{A}}_{s,t}$, $s \in [0,t] \mapsto \mathsf{L}^{\mathcal{A}}_{s,t}$ and $t \mapsto \mathsf{L}^{\mathcal{A}}_{0,t}$ are also of finite p-variation controlled by ω.

Proposition 4 (Duhamel's principle)**.** *Let $\mathcal{A} \in \mathcal{R}^p(L(\mathrm{V}^*, \mathrm{V}^*))$ (resp. $\mathcal{A} \in \mathcal{R}^p(L(\mathrm{V}, \mathrm{V}))$) and $b^* \in \mathcal{R}^p(\mathrm{V}^*)$ (resp. $b \in \mathcal{R}^p(\mathrm{V})$) with $p > 1$. The unique solution of the right (resp. left) perturbed linear RDE defined by* (34)*,* (35) *or* (36) *is*

$$y_t = y_0 \mathsf{R}^{\mathcal{A}}_{0,t} + \left(\int_0^t \mathrm{d}b^*_r \mathsf{L}^{-\mathcal{A}}_{0,r} \right) \mathsf{R}^{\mathcal{A}}_{0,t} = y_0 \mathsf{R}^{\mathcal{A}}_{0,t} + \int_0^t \mathrm{d}b^*_r \mathsf{R}^{\mathcal{A}}_{r,t}, \tag{37}$$

$$(\textit{resp. } y_t = \mathsf{L}^{\mathcal{A}}_{0,t} y_0 + \mathsf{L}^{\mathcal{A}}_{0,t} \int_0^t \mathsf{R}^{-\mathcal{A}}_{0,r} \, \mathrm{d}b_r = \mathsf{L}^{\mathcal{A}}_{0,t} y_0 + \int_0^t \mathsf{L}^{\mathcal{A}}_{r,t} \, \mathrm{d}b_r),$$

$t \in [0, T]$, where the integrals are understood as Young integrals.

Proof. We only consider the right linear RDE, the proof follows the same lines for the left ones.

The integral $\int_0^t \mathrm{d}b^*_r \mathsf{L}^{-\mathcal{A}}_{r,t}$ has to be understood as Young integral, which means that for $b_{s,t} := b_t - b_s$,

$$\int_s^t \mathrm{d}b^*_r (\mathsf{L}^{-\mathcal{A}}_{0,r}) = \lim_{n \to \infty} \sum_{i=0}^{n-1} b^*_{t^n_i, t^n_{i+1}} \mathsf{L}^{-\mathcal{A}}_{0,t^n_i}$$

for a family $\{t^n_i\}_{i=0,\dots,n}$ of partitions of $[s, t]$ whose mesh decreases to 0. Using Lemma 2,

$$\sum_{i=0}^{n-1} b^*_{t^n_i, t^n_{i+1}} \mathsf{L}^{-\mathcal{A}}_{0,t^n_i} \mathsf{R}^{\mathcal{A}}_{0,t} = \sum_{i=0}^{n-1} b^*_{t^n_i, t^n_{i+1}} \mathsf{R}^{\mathcal{A}}_{t^n_i, t}$$

so that

$$\left(\int_s^t \mathrm{d}b^*_r \mathsf{L}^{-\mathcal{A}}_{0,r} \right) \mathsf{R}^{\mathcal{A}}_{0,t} = \int_s^t \mathrm{d}b^*_r \mathsf{R}^{\mathcal{A}}_{r,t}.$$

Besides, one knows that

$$\left| \int_s^t \mathrm{d}b^*_r \mathsf{L}^{-\mathcal{A}}_{0,r} - b^*_{s,t} \mathsf{L}^{-\mathcal{A}}_{0,s} \right| \leq K \|b\|_p \|\mathsf{L}^{-\mathcal{A}}\|_p \omega(s,t)^{2p} \tag{38}$$

for some constant K that depends only on $\omega(0, T)$ and p. Let y be defined by (37). Then for $(s, t) \in \Delta_2(T)$,

$$\begin{aligned} y_t = y_0 \mathsf{R}^{\mathcal{A}}_{0,s} \mathsf{R}^{\mathcal{A}}_{s,t} + \left(\int_0^s \mathrm{d}b^*_r \mathsf{L}^{-\mathcal{A}}_{0,r} \right) \mathsf{R}^{\mathcal{A}}_{0,s} \mathsf{R}^{\mathcal{A}}_{s,t} + \left(\int_s^t \mathrm{d}b^*_r \mathsf{L}^{-\mathcal{A}}_{0,r} \right) \mathsf{R}^{\mathcal{A}}_{0,t} \\ = y_s \mathsf{R}^{\mathcal{A}}_{s,t} + \left(\int_s^t \mathrm{d}b^*_r \mathsf{L}^{-\mathcal{A}}_{0,r} - b^*_{s,t} \mathsf{L}^{-\mathcal{A}}_{0,s} \right) \mathsf{R}^{\mathcal{A}}_{0,t} + b^*_{s,t} \mathsf{L}^{-\mathcal{A}}_{0,s} \mathsf{R}^{\mathcal{A}}_{0,t}. \end{aligned}$$

Since with Lemma 2, $\mathsf{L}^{-\mathcal{A}}_{0,s} \mathsf{R}^{\mathcal{A}}_{0,t} = \mathsf{R}^{\mathcal{A}}_{s,t}$. This proves that

$$y_t - y_s \mathsf{R}^{\mathcal{A}}_{s,t} - b^*_{s,t} = \left(\int_s^t \mathrm{d}b^*_r \mathsf{L}^{-\mathcal{A}}_{0,r} - b^*_{s,t} \mathsf{L}^{-\mathcal{A}}_{0,s} \right) \mathsf{R}^{\mathcal{A}}_{0,t} + b^*_{s,t} (\mathsf{R}^{\mathcal{A}}_{s,t} - \mathsf{Id}).$$

Since $|b^*_{s,t}\mathsf{R}^{\mathcal{A}}_{s,t}| \leq \|b\|_p \|\mathsf{R}^{\mathcal{A}} - \mathsf{Id}\|_p \omega(s,t)^{2p}$ for $(s,t) \in \Delta_2(T)$.

With (38), this proves that y satisfies (34). □

Corollary 6. *Under the hypotheses of Proposition 4, the family of maps* $\Phi_{s,t} : \mathrm{V}^* \to \mathrm{V}^*$ *(resp.* $\Phi_{s,t} : \mathrm{V} \to \mathrm{V}$*) for* $(s,t) \in \Delta_2(T)$ *defined by*

$$\Phi_{s,t}(a) = y_t \text{ where } y_t = a + \int_s^t y_r \,\mathrm{d}\mathcal{A}_r + b^*_{s,t}$$

$$\text{(resp. } y_t = a + \int_s^t \mathrm{d}\mathcal{A}_r y_r + b_{s,t}\text{)}$$

forms a flow of homeomorphisms.

Proof. Again, we consider only the case of right integrals. The continuity of $\Phi_{s,t}$ with respect to a is immediate from (37), as well as the composition property: $\Phi_{r,t} \circ \Phi_{s,r} = \Phi_{s,t}$ for $(s,r,t) \in \Delta_3(T)$.

Let us set

$$\Psi_{s,t}(a) = a\mathsf{L}^{-\mathcal{A}}_{s,t} - \left(\int_s^t \mathrm{d}b^*_r \mathsf{R}^{\mathcal{A}}_{s,t}\right) \mathsf{L}^{\mathcal{A}}_{0,t}.$$

With Lemma 2, it is easily checked that $\Psi_{s,t}(\Phi_{s,t}(a)) = a$ for any $a \in \mathrm{V}^*$. □

3.4 Examples and applications

3.4.1 Controlled linear differential equation

Let $x \in \mathcal{R}^p(\mathbb{R}^d)$ with $1 \leq p < 2$ and $(A^i)_{i=1,\dots,d}$ be d square real matrices. The following linear differential equation

$$y_t = y_0 + \sum_{i=1}^d \int_0^t A^i y_s \,\mathrm{d}x^i_s,\ t \in [0,T],$$

can be seen as a linear left RDE on $\mathrm{V} = \mathbb{R}^d$ and $\mathcal{A} : [0,T] \to \mathcal{M}_{d\times d}(\mathbb{R})$ given by

$$\mathcal{A}^{j,k}_t = \sum_{i=1}^d A^i_{j,k} x^i_t,\ t \in [0,T].$$

3.4.2 Lie group values RDE

Let $\mathfrak{G}$ be a Lie matrix group with a Lie algebra $\mathfrak{g}$ [2]. Let $\mathcal{A}$ be a path in $\mathcal{R}^p(\mathfrak{g})$. We have seen that $\mathsf{A}_{s,t} = \mathsf{Id} + \mathcal{A}_t - \mathcal{A}_s$ is an almost p-resolvent.

Another way to construct an almost resolvent is to consider

$$\mathsf{B}_{s,t} = \exp(\mathcal{A}_{s,t}) \text{ with } \exp(\mathsf{M}) = \mathsf{Id} + \mathsf{M} + \frac{1}{2}\mathsf{M}^2 + \frac{1}{3!}\mathsf{M}^3 + \cdots$$

With Corollary 1, B and A gives rise to the same p-rough resolvent D (left or right).

In the context of Lie groups, considering B is more natural than A since $\mathsf{B}_{s,t}$ belongs to the Lie group $\mathfrak{G}$ for any $(s,t) \in \Delta_2(T)$. It follows immediately from Remark 3 that $\mathsf{D}_{s,t}$ belongs to the Lie group $\mathfrak{G}$.

3.4.3 Dyson and Magnus series

Let $\mathrm{V} = \mathbb{R}^d$ and let a be a continuous path from $[0,T]$ to $\mathcal{M}_{d\times d}(\mathbb{K})$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The solution to the linear ordinary differential equation

$$y'_t = \mathsf{a}(t)y_t,\ y_0 = \mathsf{Id}$$

may be expressed as the series

$$y_t = \mathsf{Id} + \sum_{n=1}^{+\infty} \int_0^t \mathrm{d}t_1 \int_0^{t_1} \cdots \int_0^{t_{n-1}} \mathrm{d}t_n \mathsf{a}(t_1)\mathsf{a}(t_2)\cdots\mathsf{a}(t_n), \tag{39}$$

called the *Dyson series*, also called *Neumann expansion* and follows from the *Picard principle* [6, 13, 20].

Let $\mathcal{A}$ be a rough path of finite p-variation controlled by ω with values in $\mathfrak{L}$.

Set $\mathsf{A}^{(1)}_{s,t} := \mathcal{A}_{s,t}$ so that $\|\mathsf{A}^{(1)}_{s,t}\| \leq \|\mathcal{A}\|_p \omega(s,t)^{1/p}$ and $\mathsf{A}^{(1)}_0 = 0$. Define recursively from $k=1$ the paths

$$\mathsf{A}^{(k+1)}_t := \int_0^t \mathrm{d}\mathcal{A}_s \mathsf{A}^{(k)}_{0,s},$$

each of the path being of finite p-variation controlled by ω with values in $\mathfrak{L}$. With (29), since $\mathsf{A}^{(k)}_0 = 0$,

$$\|\mathsf{A}^{(k+1)}\|_p \leq (1+\zeta(2p))\omega(0,T)^{1/p}\|\mathsf{A}^{(k)}\|_p\|\mathcal{A}\|_p.$$

For $\mu := (1+\zeta(2p))\omega(0,T)^{1/p}$, that

$$\|\mathsf{A}^{(k+1)}\|_p \leq \mu^k \|\mathcal{A}\|_p^{k+1},\ k = 1, 2, \dots .$$

Define

$$\mathsf{B}_t = \mathsf{Id} + \sum_{k\geq 1} \mathsf{A}^{(k)}_t. \tag{40}$$

Provided that T is small enough such that $\mu\|\mathcal{A}\|_p < 1$, the series (40) converges absolutely and

$$\sup_{t\in[0,T]} \|\mathsf{B}_t\| \leq 1 + \sum_{k\geq 1} \mu^{k-1}\|\mathcal{A}\|_p^k \omega(0,T)^{1/p} \leq 1 + \frac{1}{1+\zeta(2p)} \times \frac{\mu\|\mathcal{A}\|_p}{1-\mu\|\mathcal{A}\|_p},$$

$$\|\mathsf{B}\|_p \leq \frac{\|\mathcal{A}\|_p}{1-\mu\|\mathcal{A}\|_p}.$$

This implies that B is of finite p-variation controlled by ω.

Proposition 5. *If T is small enough so that $(1+\zeta(2p))\omega(0,T)^{1/p}\|\mathcal{A}\|_p < 1$, then B given by (40) is the solution in $\mathfrak{L}$ to the linear RDE*

$$\mathrm{d}\mathsf{Y}_t = \mathrm{d}\mathcal{A}_t \mathsf{Y}_t,\ \mathsf{Y}_0 = \mathsf{Id}.$$

Remark 10. Similarly, one may construct the solution to the right linear RDE $\mathrm{d}\mathsf{Y}_t = \mathsf{Y}_t\, \mathrm{d}\mathcal{A}_t$, $\mathsf{Y}_0 = \mathsf{Id}$.

Remark 11. Consider for $\mathfrak{L}$ the tensor space $\mathrm{T}_\infty(\mathbb{R}^d)$ and $\mathcal{A}_{s,t} = x_t - x_s$ for a path of finite p-variation controlled by ω with values in $\mathbb{R}^d$. As shown by K.T. Chen in [13, 14], the solution to $\mathrm{d}\mathsf{Y}_t = \mathsf{Y}_t\, \mathrm{d}\mathcal{A}_t$ takes its values in $\mathrm{T}_\infty(\mathbb{R}^d)$ and corresponds to the Chen series associated to x.

Proof. Since B is of finite p-variation controlled by ω and B_t is an absolutely converging series,

$$\mathsf{Id} + \int_0^t \mathrm{d}\mathcal{A}_s \mathsf{B}_s = \mathsf{Id} + \sum_{k\geq 1} \mathsf{A}_t^{(k)} = \mathsf{B}_t,$$

which allows one to conclude. □

For a continuous path a with values in $\mathfrak{L} = \mathcal{M}_{d\times d}(\mathbb{K})$ and $\mathcal{A}_t = \int_0^t \mathsf{a}(s)\,\mathrm{d}s$, (40) is nothing more than the Dyson series (39).

Assume from now that $\mathrm{V} = \mathbb{K}^d$ and identify $\mathfrak{L} = L(\mathrm{V},\mathrm{V})$ with $\mathcal{M}_{d\times d}(\mathbb{K})$.

Then, a linear equation driven by $\mathcal{A} \in \mathcal{R}^p(\mathcal{M}_{d\times d}(\mathbb{K}))$ may indeed be identified with a linear equation controlled by a geometric p-rough path $\mathbf{x} = (\mathbf{x}_t^{i,j})_{i,j=1,\dots,d,t\in[0,T]}$ with values in $\mathrm{T}_1(\mathbb{K}^{d\times d})$ and for $A^{i,j}$ defined by $A^{i,j} = (\delta_{i,j}^{k,\ell})_{k,\ell=1,\dots,d}$, we have

$$\mathcal{A}_{s,t} = \sum_{i,j=1}^d A^{i,j}\mathbf{x}_{s,t}^{i,j},\ (s,t)\in\Delta_2(T).$$

For $\mathcal{A}_t = \int_0^t \mathsf{a}(t)\,\mathrm{d}t$ and σ a permutation of $\{1,\dots,k\}$, set

$$\begin{aligned}\int_0^t [\,\mathrm{d}\mathcal{A}_{t_{\sigma(k)}}, [\cdots, [\,\mathrm{d}\mathcal{A}_{t_{\sigma(2)}},\, \mathrm{d}\mathcal{A}_{t_{\sigma(1)}}]\cdots]]\\ := \int_0^t\int_0^{t_k}\cdots\int_0^{t_2} [\mathsf{a}(t_{\sigma(k)}), [\cdots, [\cdots[\mathsf{a}(t_{\sigma(2)}), \mathsf{a}(t_{\sigma(1)})]]\cdots]\,\mathrm{d}t_1\cdots\mathrm{d}t_k. \quad (41)\end{aligned}$$

Now, it is easily shown that $\mathcal{A}\in\mathcal{R}^p(\mathcal{M}_{d\times d}(\mathbb{K}))$, is the limit in q-variation of piecewise linear paths $\mathcal{A}^n$ for any $q<p$. Let us write $\mathcal{A}_t^n = \int_0^t \mathsf{a}^n(s)\,\mathrm{d}s$. By continuity of the Young integral, we define the left hand side of (41) as the limit of the right hand side of (41) with a replaced by a^n.

Theorem 3 may then be rewritten the following way.

Proposition 6. *When $\mathcal{A} \in \mathcal{R}^p(\mathcal{M}_{d\times d}(\mathbb{K}))$, $1 \le p < 2$, then for T small enough (in function of ω, $\|\mathcal{A}\|_p$ and p),*

$$\mathsf{B}_t = \exp\left(\sum_{k\ge1}\sum_{\sigma\in\mathfrak{S}_k} \frac{(-1)^{e(\sigma)}}{k^2\binom{k-1}{e(\sigma)}} \int_0^t [\,\mathrm{d}\mathcal{A}_{t_{\sigma(k)}}, [\cdots, [\,\mathrm{d}\mathcal{A}_{t_{\sigma(2)}}, \mathrm{d}\mathcal{A}_{t_{\sigma(1)}}]\cdots]]\right)$$

for $0 \le t \le T$.

Remark 12. One recovers the Magnus formula [6]. In particular, the first terms are

$$\mathsf{B}_t = \exp\left(\mathcal{A}_{0,t} + \frac{1}{2}\int_0^t \mathrm{d}\mathcal{A}_s\mathcal{A}_{0,s} - \frac{1}{2}\int_0^t \mathcal{A}_{0,s}\,\mathrm{d}\mathcal{A}_s + \cdots\right).$$

Remark 13. More generally, even in the case of differentiable paths $\mathcal{A}$, an interesting question concerns the maximal time interval on which the Magnus formula is defined. See [6,40] for insights and sharp results about this problem in the case of differentiable paths $\mathcal{A}$.

4 Linear and linear perturbed equations when $2 \le p < 3$

4.1 Perturbation of linear RDE

In this section we adapt the ideas of the previous section.

Let $\mathcal{A} \in \mathcal{R}^p(L(\mathrm{V},\mathrm{V}))$ and let us assume that there exists two families of operators $\{\mathsf{A}^{(1)}_{s,t}\}_{(s,t)\in\Delta_2(T)}$ and $\{\mathsf{A}^{(2)}_{s,t}\}_{(s,t)\in\Delta_2(T)}$ such that for some $\theta > 1$,

$$\mathsf{A}^{(1)}_{s,t} = \mathcal{A}_t - \mathcal{A}_s,\ (s,t) \in \Delta_2(T)$$
$$|\mathsf{A}^{(2)}_{s,t} - \mathsf{A}^{(2)}_{s,r} - \mathsf{A}^{(2)}_{r,t} - \mathsf{A}^{(1)}_{s,r}\mathsf{A}^{(1)}_{r,t}| \le C\omega(s,t)^\theta,\ (s,r,t) \in \Delta_3(T),$$
$$\|\mathsf{A}^{(1)}_{s,t}\| \le C\omega(s,t)^{1/p},\ \|\mathsf{A}^{(2)}_{s,t}\| \le C\omega(s,t)^{2/p}.$$

Set $\mathbf{A}_{s,t} := \mathsf{A}^{(1)}_{s,t} + \mathsf{A}^{(2)}_{s,t}$ and $\mathsf{A}_{s,t} := \mathsf{Id} + \mathbf{A}_{s,t}$.

Definition 4. When it exists, we call $\mathbf{A}$ a p-right lift of $\mathcal{A}$.

The proof of the following lemma is straightforward.

Lemma 4. *The family $\{\mathsf{A}_{s,t}\}_{(s,t)\in\Delta_2(T)}$ is an almost right resolvent.*

Let us set

$$\mathcal{B}_{s,t} := -\mathsf{A}^{(1)}_{s,t} - \mathsf{A}^{(2)}_{s,t} + \mathsf{A}^{(1)}_{s,t}\mathsf{A}^{(1)}_{s,t} \text{ and } \mathsf{B}_{s,t} := \mathsf{Id} + \mathcal{B}_{s,t}.$$

Lemma 5. *The family* $\{\mathsf{B}_{s,t}\}_{(s,t)\in\Delta_2(T)}$ *is an almost left resolvent.*

Proof. We have

$$\mathsf{B}_{s,t} = \mathsf{Id} - \mathsf{A}^{(1)}_{s,r} - \mathsf{A}^{(1)}_{r,t} - \mathsf{A}^{(2)}_{s,r} - \mathsf{A}^{(2)}_{r,t} + \mathsf{A}^{(1)}_{s,r}\mathsf{A}^{(1)}_{s,r} + \mathsf{A}^{(1)}_{r,t}\mathsf{A}^{(1)}_{r,t} + \mathsf{A}^{(1)}_{r,t}\mathsf{A}^{(1)}_{s,r} + \sigma_{s,r,t}$$

with $\|\sigma_{s,r,t}\| \le C\omega(s,t)^\theta$ so that

$$\|\mathsf{B}_{s,t} - \mathsf{B}_{r,t}\mathsf{B}_{s,r}\| \le C\omega(s,t)^{\theta\wedge 3p}.$$

This proves that B is an almost left-resolvent since $\theta > 1$ and $3p > 1$. □

Proposition 7. *For any* $t \ge 0$, $\mathsf{R}^{\mathsf{A}}_{0,t}\mathsf{L}^{\mathsf{B}}_{0,t} = \mathsf{L}^{\mathsf{B}}_{0,t}\mathsf{R}^{\mathsf{A}}_{0,t} = \mathsf{Id}$.

Proof. Let us remark that for $(s,t) \in \Delta_2(T)$,

$$\|\mathsf{R}^{\mathsf{A}}_{s,t}\mathsf{L}^{\mathsf{B}}_{s,t} - (\mathsf{Id} + \mathsf{A}^1_{s,t} + \mathsf{A}^2_{s,t})(\mathsf{Id} - \mathsf{A}^1_{s,t} - \mathsf{A}^2_{s,t} + \mathsf{A}^1_{s,t}\mathsf{A}^1_{s,t})\| \le C\omega(s,t)^\theta$$

since both $\|\mathsf{R}^{\mathsf{A}}_{s,t} - \mathsf{Id} - \mathsf{A}^1_{s,t} - \mathsf{A}^2_{s,t}\|$ and $\|\mathsf{L}^{\mathsf{B}}_{s,t} - \mathsf{Id} + \mathsf{A}^1_{s,t} + \mathsf{A}^2_{s,t} - \mathsf{A}^1_{s,t}\mathsf{A}^1_{s,t}\|$ are both smaller than $C\omega(s,t)^\theta$ for $\theta > 1$. On the other hand,

$$(\mathsf{Id} + \mathsf{A}^1_{s,t} + \mathsf{A}^2_{s,t})(\mathsf{Id} - \mathsf{A}^1_{s,t} - \mathsf{A}^2_{s,t} + \mathsf{A}^1_{s,t}\mathsf{A}^1_{s,t}) = \sigma^1(s,t) + \mathsf{Id}$$

with $\|\sigma^1(s,t)\| \le \omega(s,t)^{3p}$ and $3p > 1$. Besides,

$$\mathsf{R}^{\mathsf{A}}_{0,t}\mathsf{L}^{\mathsf{B}}_{0,t} = \mathsf{R}^{\mathsf{A}}_{0,s}\mathsf{R}^{\mathsf{A}}_{s,t}\mathsf{L}^{\mathsf{B}}_{r,t}\mathsf{L}^{\mathsf{B}}_{0,r} = \mathsf{R}^{\mathsf{A}}_{0,s}(\mathsf{Id} + \sigma^2(s,t))\mathsf{L}^{\mathsf{B}}_{0,s}$$

with $|\sigma_2(s,t)| \le C\omega(s,t)^{3p\wedge\theta}$. This way, $\mathsf{R}^{\mathsf{A}}_{0,t}\mathsf{L}^{\mathsf{B}}_{0,t}$ is of finite $3p \wedge \theta$-variation controlled by ω and is constant. To conclude, it remains to remark that $\mathsf{L}^{\mathsf{B}}_{0,0} = \mathsf{R}^{\mathsf{A}}_{0,0} = \mathsf{Id}$. Note that both $\mathsf{L}^{\mathsf{B}}_{0,t}$ and $\mathsf{R}^{\mathsf{A}}_{0,t}$ are invertible. □

Definition 5 (Linear RDE)**.** Let $\mathcal{A} \in \mathcal{R}^p(L(\mathrm{V}^*, \mathrm{V}^*))$ (resp. $\mathcal{R}^p(L(\mathrm{V}, \mathrm{V}))$) which admits a p-rough lift $\mathbf{A}$. We say that a continuous path y of finite p-variation controlled by ω is the solution to the right (resp. left) *linear RDE* if there exists some constants $C > 0$ and $\theta > 1$ such that for any $0 \le s \le t \le T$,

$$|y_t - y_s\mathsf{A}_{s,t}| \le C\omega(s,t)^\theta \text{ (resp. } |y_t - \mathsf{A}_{s,t}y_s| \le C\omega(s,t)^\theta). \tag{42}$$

We also denote y as the solution to $y_t = y_s + \int_s^t y_r \,\mathrm{d}\mathbf{A}_r$ (resp. $y_t = y_s + \int_s^t \mathrm{d}\mathbf{A}_r y_r$) .

Proposition 8. *Let* $\mathcal{A} \in \mathcal{R}^p(L(\mathrm{V}, \mathrm{V}))$ *(resp.* $\mathcal{R}^p(L(\mathrm{V}, \mathrm{V}))$*) having a p-rough right (resp. left) lift* $\mathbf{A}$. *The unique solution of the right (resp. left) linear RDE is* $y_t = y_0\mathsf{R}^{\mathsf{A}}_{0,t}$, *(resp* $y_t = \mathsf{L}^{\mathsf{A}}_{0,t}y_0$*),* $t \in [0,T]$. *Thus,* R^{A} *(resp.* L^{A}*) define some a flow of homeomorphisms.*

Proof. Let y be a path in $\mathcal{R}^p(\mathrm{V}^*)$ satisfying (42) and set $z_t = y_t \mathsf{L}^{\mathsf{B}}_{0,t}$, $t \in [0,T]$. Then

$$z_t = (y_t - y_s \mathsf{R}^{\mathsf{A}}_{s,t})\mathsf{L}^{\mathsf{B}}_{0,t} + y_s \mathsf{R}^{\mathsf{A}}_{s,t}\mathsf{L}^{\mathsf{B}}_{0,t}.$$

With Proposition 7, $y_s \mathsf{R}^{\mathsf{A}}_{s,t}\mathsf{L}^{\mathsf{B}}_{0,t} = y_s \mathsf{L}^{\mathsf{B}}_{0,s} = z_s$. With (3), $|z_t - z_s| \leq C\omega(s,t)^\theta$ with $\theta > 1$ so that z is constant and $z_t = y_0$ for $t \in [0,T]$ since $\mathsf{L}^{\mathsf{B}}_{0,0} = \mathsf{Id}$. Again with Proposition 7, This proves that $y_t = y_0 \mathsf{R}^{\mathsf{A}}_{0,t}$.

On the other hand, for $y_t := y_0 \mathsf{R}^{\mathsf{A}}_{0,t}$, we have

$$y_t - y_s \mathsf{A}_{s,t} = y_s(\mathsf{R}^{\mathsf{A}}_{s,t} - \mathsf{A}_{s,t})$$

and from the construction of R^{A} from A in Theorem 1, (42) is satisfied for y which is then a solution to the right linear RDE.

That R^{A} and L^{A} define flow of homeomorphisms is immediate from Proposition 7. □

Definition 6 (Rough lift)**.** Let $b \in \mathcal{R}^p(\mathrm{V}^*)$ (resp. $b \in \mathcal{R}^p(\mathrm{V})$). We say that b have a *p-rough right (resp. left) lift* with respect to $\mathbf{A}$, denoted by $\mathbf{b} = \mathbf{b}^{(1)} + \mathbf{b}^{(2)}$ is a function from $\Delta_2(T) \to \mathrm{V}^*$ (resp. V) satisfying

$$\begin{gathered}
\mathbf{b}^{(1)}_{s,t} = b_t - b_s \text{ for } (s,t) \in \Delta_2(T),\\
\|\mathbf{b}^{(2)}\|_{p/2} := \sup_{(s,t)\in\Delta_2(T)} \frac{|\mathbf{b}^{(2)}_{s,t}|}{\omega(s,t)^{p/2}} < +\infty,\\
\mathbf{b}^{(2)}_{s,t} = \mathbf{b}^{(2)}_{s,r} + \mathbf{b}^{(2)}_{r,t} + \mathbf{b}^{(1)}_{s,r}\mathbf{A}^{(1)}_{r,t},\ \forall (s,r,t) \in \Delta_3(T),\\
(\text{resp. } \mathbf{b}^{(2)}_{s,t} = \mathbf{b}^{(2)}_{s,r} + \mathbf{b}^{(2)}_{r,t} + \mathbf{A}^{(1)}_{r,t}\mathbf{b}^{(1)}_{s,r}).
\end{gathered}$$

Definition 7 (Perturbed linear RDE)**.** A path $y \in \mathcal{R}^p(\mathrm{V}^*)$ is said to be a solution to the right *perturbed linear RDE* by b having a p-rough lift right (resp. left) $\mathbf{b}$ with respect to $\mathbf{A}$ if there exists some constants $C > 0$ and $\theta > 1$ such that

$$\begin{gathered}
|y_t - y_s \mathsf{R}^{\mathsf{A}}_{s,t} - \mathbf{b}_{s,t}| \leq C\omega(s,t)^\theta,\ \forall (s,t) \in \Delta_2(T), \qquad (43)\\
(\text{resp. } |y_t - \mathsf{L}^{\mathsf{A}}_{s,t} y_s - \mathbf{b}_{s,t}| \leq C\omega(s,t)^\theta).
\end{gathered}$$

We also denote y as the solution to $y_t = y_s + \int_s^t y_r \,\mathrm{d}\mathbf{A}_r + b_{s,t}$ (resp.$y_t = y_s + \int_s^t \mathrm{d}\mathbf{A}_r y_r + b_{s,t}$).

Remark 14. In this definition, one may change R^{A} and L^{A} by A in (43), since for any $(s,t) \in \Delta_2(T)$, $\|\mathsf{A}_{s,t} - \mathsf{R}^{\mathsf{A}}_{s,t}\| \leq C\omega(s,t)^\theta$, and $\|\mathsf{A}_{s,t} - \mathsf{L}^{\mathsf{A}}_{s,t}\| \leq C\omega(s,t)^\theta$.

Lemma 6. *Let $\mathcal{A} \in \mathcal{R}^p(L(\mathrm{V}^*,\mathrm{V}^*))$ (resp. $\mathcal{A} \in \mathcal{R}^p(L(\mathrm{V},\mathrm{V}))$) which admits a p-rough right (resp. left) lift $\mathbf{A}$. Let $b \in \mathcal{R}^p(\mathrm{V})$ which admits a p-rough right (resp. left) with respect to $\mathbf{A}$. Let μ be the functional defined for $u \in [0,T]$ by*

$$\mu(u)_{s,r} := \mathbf{b}_{s,r}\mathsf{R}^{\mathsf{A}}_{r,u}, (\textit{resp. } \mu(u)_{s,r} := \mathsf{L}^{\mathsf{A}}_{r,u}\mathbf{b}_{s,r}),\ (s,r) \in \Delta_2(u),$$

Then there exists, C_1 and C_2 depending only on $\|\mathcal{A}\|_p$, $\|\mathbf{A}^{(2)}\|_{p/2}$, $\omega(0,T)$, $\theta > 1$ and p such that

$$|\mu(u)_{s,t}| \leq C_1(\|b\|_p + \|\mathbf{b}^{(2)}\|_{p/2})\omega(s,t)^{1/p},\ \forall (s,t) \in \Delta_2(u),$$
$$|\delta_{s,r,t}\mu(u)| \leq C_2(\|b\|_p + \|\mathbf{b}^{(2)}\|_{p/2})\omega(s,t)^{\theta},\ (s,r,t) \in \Delta_3(u),\ u \in [0,T],$$

where $\delta_{s,r,t}\mu(u) = \mu(u)_{s,t} - \mu(u)_{s,r} - \mu(u)_{r,t}$.

Proof. For $(s,r,t) \in \Delta_3(u)$,

$$\delta_{s,r,t}\mu(u) = (b_r - b_s + \mathbf{b}^{(2)}_{s,t} - \mathbf{b}^{(2)}_{r,t})\mathsf{R}^{\mathsf{A}}_{t,u} - (b_r - b_s + \mathbf{b}^{(2)}_{s,r})\mathsf{R}^{\mathsf{A}}_{r,u}.$$

But $\mathsf{R}^{\mathsf{A}}_{r,u} = \mathsf{R}^{\mathsf{A}}{}_{r,t}\mathsf{R}^{\mathsf{A}}{}_{t,u}$ and

$$\delta_{s,r,t}\mu(u) = [b_r - b_s + \mathbf{b}^{(2)}_{s,t} - \mathbf{b}^{(2)}_{r,t} - (b_r - b_s + \mathbf{b}^{(2)}_{s,r})\mathsf{R}^{\mathsf{A}}_{r,t}]\mathsf{R}^{\mathsf{A}}_{t,u}.$$

Using the definition of $\mathbf{b}^{(2)}$, we derive

$$\delta_{s,r,t}\mu(u) = [b_{s,r}(\mathsf{Id} - \mathbf{A}^{(1)}_{r,t} - \mathsf{R}^{\mathsf{A}}_{r,t}) + \mathbf{b}^{(2)}_{s,r}(\mathsf{Id} - \mathsf{R}^{\mathsf{A}}_{r,t})]\mathsf{R}^{\mathsf{A}}_{t,u}.$$

But from the very construction of R^{A}, for some constant C_5 depending only on $\|\mathbf{A}^{(1)}\|$, $\|\mathbf{A}^{(2)}\|$, $\omega(0,T)$, θ and p,

$$\|\mathsf{R}^{\mathsf{A}}_{r,t} - \mathsf{Id} - \mathbf{A}^{(1)}_{r,t} - \mathbf{A}^{(2)}_{r,t}\| \leq C_5\omega(r,t)^{\theta},\ \theta > 1,$$

since for some constants C_3 and C_4, $\|\mathbf{A}^{(1)}_{r,t}\| \leq C_3\omega(r,t)^{1/p}$, $\|\mathbf{A}^{(2)}_{r,t}\| \leq C_4\omega(r,t)^{2/p}$. This is sufficient to prove the result. $\square$

From the control we get on $\mu(u)$, the sewing lemma [21] shows that there exists a path $\{\widehat{\mu}(u)_t\}_{t\in[0,u]}$ and a constant K depending only on $\omega(0,T)$, θ and p such that

$$|\mu(u)_{s,t} - \widehat{\mu}(u)_t + \widehat{\mu}(u)_s| \leq KC_2(\|b\|_p + \|\mathbf{b}^{(2)}\|_{p/2})\omega(s,t)^{\theta},\ (s,t) \in \Delta_2(u).$$

This path is denoted by $\widehat{\mu}(u)_t := \int_0^t \mathrm{d}^{\mathrm{b}}\mathbf{b}_r\mathsf{R}^{\mathsf{A}}_{r,u}$ for $t \in [0,u]$. Here the superscript b on d^{b} stands for "backward integration". Using the fact that R^{A} is a p-rough resolvent,

$$\mu(u)_{s,r} = \mu(v)_{s,r}\mathsf{R}^{\mathsf{A}}_{v,u},\ u < v$$

and then from the uniqueness in the Sewing lemma [21],

$$\int_0^r \mathrm{d}^{\mathrm{b}}\mathbf{b}_s\mathsf{R}^{\mathsf{A}}_{s,u} = \left(\int_0^r \mathrm{d}^{\mathrm{b}}\mathbf{b}_s\mathsf{R}^{\mathsf{A}}_{s,v}\right)\mathsf{R}^{\mathsf{A}}_{v,u},\ \forall (r,v,u) \in \Delta_3(T). \tag{44}$$

Proposition 9 (Duhamel's principle)**.** *Let $\mathcal{A} \in \mathcal{R}^p(L(\mathrm{V}^*, \mathrm{V}^*))$ which admits a p-rough right lift $\mathbf{A}$. Let $b \in \mathcal{R}^p(\mathrm{V}^*)$ which admits a p-right lift with respect to $\mathbf{A}$. The unique solution of the perturbed linear RDE*

$$y_t = y_0 + \int_0^t y_s \, \mathrm{d}\mathbf{A}_s + b_{0,t}$$

is

$$y_t = y_0 \mathsf{R}^{\mathsf{A}}_{0,t} + \int_0^t \mathrm{d}^{\mathrm{b}} b_r \mathsf{R}^{\mathsf{A}}_{r,t}.$$

Besides, there exists a constant C depending only on $\|\mathcal{A}\|_p$, $\|\mathbf{A}^{(2)}\|_{p/2}$, $\omega(0,T)$ and p such that

$$\|y\|_p \le C(|y_0| + \|b\|_p + \|\mathbf{b}^{(2)}\|_{p/2}).$$

A similar result holds for left linear RDE.

Proof. The only consider the right linear RDE, the proof follows the same lines is the left side.

Uniqueness of the solution of the linear perturbed RDE follows from the uniqueness of the linear RDE.

If $z_t := \int_0^t \mathrm{d}^{\mathrm{b}} \mathbf{b}_r \mathsf{R}^{\mathsf{A}}_{r,t}$, then with (44),

$$\int_s^t \mathrm{d}^{\mathrm{b}} \mathbf{b}_r \mathsf{R}^{\mathsf{A}}_{r,t} - \mu(t)_{s,t} = z_t - z_s \mathsf{R}^{\mathsf{A}}_{s,t} - \mathbf{b}_{s,t}.$$

On the other hand, from the sewing lemma,

$$\left| \int_s^t \mathrm{d}^{\mathrm{b}} \mathbf{b}_r \mathsf{R}^{\mathsf{A}}_{r,t} - \mu(t)_{s,t} \right| \le C\omega(s,t)^\theta, \ s \in [0,t].$$

This proves that z is solution to the perturbed linear RDE with $z_0 = 0$. On the other hand, we have seen in Proposition 8 that $a\mathsf{R}^{\mathsf{A}}_{0,\cdot}$ is solution to the linear RDE $z_t = a + \int_0^t z_r \, \mathrm{d}\mathbf{A}_r$. By linearity, this proves the result. □

Corollary 7. *Let $\Phi_{s,t}$ be the map from V^* to V^* defined by $\Phi_{s,t}(a) := y_t$ where y is the solution to $y_u = a + \int_s^u y_r \, \mathrm{d}\mathbf{A}_r + b_{s,u}$, $u \ge s$, so that $\Phi_{s,t}(a) = a\mathsf{R}^{\mathsf{A}}_{s,t} + \int_s^t \mathrm{d}^{\mathrm{b}} \mathbf{b}_r \mathsf{R}^{\mathsf{A}}_{r,t}$. Then $\Phi_{s,t}(a)$ defines a flow of homeomorphisms from V^* to V^*.*

A similar result holds for left linear RDE.

Proof. It is immediate that $\Phi_{s,t}$ is continuous on V^*, and that $\Phi_{r,t} \circ \Phi_{s,r} = \Phi_{s,t}$ with (44). Let us define

$$\Psi_{s,t}(a) = a\mathsf{L}^{\mathsf{B}}_{s,t} - \left(\int_s^t \mathrm{d}^{\mathrm{b}} \mathbf{b}_r \mathsf{R}^{\mathsf{A}}_{r,t} \right) \mathsf{L}^{\mathsf{B}}_{s,t}.$$

Hence,

$$\Phi_{s,t}(\Psi_{s,t}(a)) = a$$

so that $(\Phi_{s,t})_{(s,t)\in\Delta_2(T)}$ defines a flow of homeomorphisms from V^* to V^*. □

To end with, let us note that if the perturbation b is regular enough, then there is no need to consider a lift of b.

Proposition 10. *Let $b \in \mathcal{R}^q(\mathrm{V}^*)$ with $1/p + 1/q > 1$. Then with $\mathbf{b}^{(2)}_{s,t} := \int_s^t b_{s,r}\,\mathrm{d}\mathcal{A}_r$, $\mathbf{b}_{s,t} = b_{s,t} + \mathbf{b}^{(2)}_{s,t}$ defines a right p-rough lift of b with*

$$|\mathbf{b}^{(2)}_{s,t}| \leq K\|b\|_q\|\mathcal{A}\|_p\omega(s,t)^{1/p+1/q}. \tag{45}$$

Besides,

$$\int_0^t \mathrm{d}^{\mathrm{b}}\mathbf{b}_r \mathsf{R}^{\mathsf{A}}_{r,t} = \int_0^t \mathrm{d}b_r \mathsf{R}^{\mathsf{A}}_{r,t}$$

where the integral in the right hand side has to be understood as a Young integral.

A similar result holds for left linear RDE.

Proof. From the condition $1/p+1/q > 1$, $\int_s^t b_{s,r}\,\mathrm{d}\mathcal{A}_r$ is well defined as a rough integral and (45) is immediate. Besides, $\int_0^t \mathrm{d}b_r\mathsf{R}^{\mathsf{A}}_{r,t}$ is well defined as a Young integral and is associated to the almost additive function $\nu(t)_{s,r} = b_{s,r}\mathsf{R}^{\mathsf{A}}_{s,t}$. But

$$\mu(t)_{s,r} - \nu(t)_{s,r} = b_{s,r}(\mathsf{Id} - \mathsf{R}^{\mathsf{A}}_{s,r})\mathsf{R}^{\mathsf{A}}_{r,t} + \mathbf{b}^{(2)}_{s,r}\mathsf{R}^{\mathsf{A}}_{r,t}.$$

But $\|\mathsf{Id} - \mathsf{R}^{\mathsf{A}}_{s,r}\| \leq C\omega(s,r)^{1/p}$ and $\|\mathbf{b}^{(2)}_{s,r}\| \leq C\omega(s,r)^{1/p+1/q}$. With (45),

$$|\mu(t)_{s,r} - \nu(t)_{s,r}| \leq C\omega(s,r)^{1/p+1/q}.$$

This is sufficient to prove the equality of the corresponding integrals. □

4.2 Integration on the Heisenberg groups, $2 \leq p < 3$

Let $\mathbf{x}$ be a weak geometric rough path (see however Remark 7 on non-geometric rough paths) with values in $\mathrm{T}_2(\mathbb{R}^d)$ of finite p-variation controlled by ω, $2 \leq p < 3$.

Let $A_1, \dots, A_d$ be matrices in the Heisenberg algebra

$$\mathfrak{h} := \left\{ \begin{bmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix} \,\middle|\, (a,b,c) \in \mathbb{R}^3 \right\}.$$

The space $\mathfrak{h}$ is the Lie algebra of the Heisenberg group

$$\mathfrak{H} := \left\{ \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \,\middle|\, (a,b,c) \in \mathbb{R}^3 \right\} = \exp(\mathfrak{h}).$$

The product of 3 matrices in $\mathfrak{h}$ is equal to 0 so that $\mathfrak{h}$ is a simply connected nilpotent Lie algebra of step 2.

Set

$$\mathbf{a}^{(1)}_{s,t} := \sum_{i=1}^{d} A_i \mathbf{x}^{(1),i}_{s,t} \text{ and } \mathbf{a}^{(2)}_{s,t} := \sum_{1\leq i<j\leq d} [A_i, A_j]\overline{\mathbf{x}}^{(2),i,j}_{s,t},\ (s,t)\in\Delta_2(T),$$

where $\overline{\mathbf{x}}^{(2),i,j} := \frac{1}{2}(\mathbf{x}^{(2),i,j} - \mathbf{x}^{(2),j,i})$, the anti-symmetric part of $\mathbf{x}^{(2)}$. Set also

$$\mathbf{A}^{(1)}_{s,t} := \sum_{i=1}^{d} A_i \mathbf{x}^{(1),i}_{s,t} \text{ and } \mathbf{A}^{(2)}_{s,t} := \sum_{i,j}^{d} A_i A_j \mathbf{x}^{(2),i,j}_{s,t},\ (s,t)\in\Delta_2(T),$$

and $\mathsf{A}_{s,t} := \mathsf{Id} + \mathbf{A}^{(1)}_{s,t} + \mathbf{A}^{(2)}_{s,t}$. Note that $\mathsf{A}_{s,t} = \exp(\mathbf{a}^{(1)}_{s,t} + \mathbf{a}^{(2)}_{s,t})$. Since

$$\mathbf{x}^{(2),i,j}_{s,t} = \mathbf{x}^{(2),i,j}_{s,r} + \mathbf{x}^{(2),i,j}_{r,t} + \mathbf{x}^{(1),i}_{s,r}\mathbf{x}^{(1),j}_{r,t},\ \forall (s,r,t)\in\Delta_3(T),\ \forall i,j = 1,\dots,d,$$

it follows that $\mathsf{A}_{s,r}\mathsf{A}_{r,t} = \mathsf{A}_{s,t}$ for all $(s,r,t)\in\Delta_3(T)$. Thus, A is a lift of $\mathbf{A}^{(1)}_{s,t}$ and A is a right p-rough resolvent.

This means that the solution of

$$y_t = y_s + \int_0^t y_s \,\mathrm{d}\mathcal{A}_s,\ \mathcal{A}_t = \sum_{i=1}^{d} A_i x^{1,i}_t \tag{46}$$

is given by

$$y_t = y_0 \mathsf{A}_{0,t} = y_0 \exp(\mathbf{a}^{(1)}_{s,t} + \mathbf{a}^{(2)}_{s,t})$$

and y_t belongs to the Heisenberg group $\mathfrak{H}$. This implies that one may solve (46) by solving the ODE

$$\dot{z}_{(s,t)}(\tau) = z_{(s,t)}(\tau)(\mathbf{a}^{(1)}_{s,t} + \mathbf{a}^{(2)}_{s,t}),\ \tau\in[0,1],\ z_0 = y_s$$

and setting $y_t = z_{(s,t)}(1)$.

4.3 Application to rough differential equations

Let U, V and W be Banach spaces and W is assumed to be finite-dimensional. Let $\mathbf{x}\in\mathcal{R}^p(\mathrm{T}_2(\mathrm{U}))$ be a rough path of finite p-variation controlled by ω. Let $y\in\mathcal{R}^p(\mathrm{V})$ such that there exists a family $\{y\ltimes x_{s,t}\}_{(s,t)\in\Delta_2(T)}$ with values in $\mathrm{V}\otimes\mathrm{U}$ for which

$$y\ltimes x_{s,t} = y\ltimes x_{s,r} + y\ltimes x_{r,t} + y_{s,r}\times x_{r,t} \text{ and } |y\ltimes x_{s,t}| \leq C\omega(s,t)^{p/2}$$

for all $(s,r,t)\in\Delta_3(T)$.

Let us consider a function $f:\mathrm{V}\to L(\mathrm{W}\otimes\mathrm{U},\mathrm{W})$ such that f is bounded with a bounded derivative ∇f, which is γ-Hölder continuous, $2+\gamma>p$.

Our first aim is to consider the solution to the differential equation

$$z_t = a + \int_0^t f(y_s) z_s \, \mathrm{d}\mathbf{x}_s. \tag{47}$$

If $\mathrm{U} = \mathbb{R}^d$, $\mathrm{V} = \mathbb{R}^n$ and $\mathrm{W} = \mathbb{R}^m$ and $\mathbf{x}$ is a smooth rough path living above x, by an equation of type (47), we mean

$$z_t^i = a^i + \sum_{\substack{k=1,\dots,d\\ j=1,\dots,n}} \int_0^t f_k^{i,j}(y_s) z_s^j \, \mathrm{d}x_s^k, \ i = 1, \dots, m.$$

In the sense of Davie, a solution of (47) is sought as satisfying for some constants $C \geq 0$ and $\theta > 1$, with $\mathbf{x}_{s,t}^{(1)} := x_t - x_s$ and $\mathbf{x}_{s,t}^{(2)} := \int_s^t [x_r - x_s] \otimes dx_r$,

$$\begin{aligned}\sup_{i=1,\dots,m} \Bigg| z_t^i - z_s^i - \sum_{\substack{k=1,\dots,d\\ j=1,\dots,n}} f_k^{i,j}(y_s) z_s^j \mathbf{x}_{s,t}^{(1),k} - \sum_{\substack{k=1,\dots,d\\ j=1,\dots,n\\ r=1,\dots,n}} \frac{\partial f_k^{i,j}}{\partial y^r}(y_s) z_s^j (y \ltimes x)_{s,t}^{r,k} \\ - \sum_{\substack{k=1,\dots,d\\ j=1,\dots,n\\ q=1,\dots,d\\ p=1,\dots,n}} f_k^{i,j}(y_s) f_q^{j,p}(y_s) z_s^p \mathbf{x}_{s,t}^{(2),q,k} \Bigg| \leq C\omega(s,t)^\theta, \ \forall (s,t) \in \Delta_2(T),\end{aligned}$$

which we write under the more compact form

$$|z_t - z_s - f(y_s) z_s \mathbf{x}_{s,t}^{(1)} - \nabla f(y_s) z_s y \ltimes x_{s,t} - F(y_s) z_s \mathbf{x}_{s,t}^{(2)}| \leq C\omega(s,t)^\theta, \tag{48}$$

with $F_{k,\ell}(y_s) = f_k(y_s) f_\ell(y_s)$ is a matrix in $\mathbb{R}^m$ for $k, \ell = 1, \dots, d$, and then F may be identified with a map from V to $L(\mathrm{W} \otimes \mathrm{U} \otimes \mathrm{U}, \mathrm{W})$.

Proposition 11. *Let $\mathcal{A}_t = \int_0^t f(y_s) \, \mathrm{d}\mathbf{x}_s$ be the path $\mathcal{R}^p(L(\mathrm{W},\mathrm{W}))$ by the rough integral $\mathcal{A}_t z = \left(\int_0^t f(y_s) z \, \mathrm{d}\mathbf{x}_s\right)^{(1)} \in \mathcal{R}^p(\mathrm{W})$ for any $z \in \mathrm{W}$. Then there exists a p-left rough lift B of $\mathcal{A}$.*

Proof. For $(s,t) \in \Delta_2(T)$, let $\mathsf{A}_{s,t}$ be the linear operator in $L(\mathrm{W},\mathrm{W})$ defined by

$$\mathsf{A}_{s,t} z = z + f(y_s)[z] \mathbf{x}_{s,t}^{(1)} + \nabla f(y_s)[z] y \ltimes x_{s,t} + F(y_s)[z] \mathbf{x}_{s,t}^{(2)}, \ z \in \mathrm{W}.$$

Then

$$\begin{aligned}\mathsf{A}_{r,t} \mathsf{A}_{s,r} z = z &+ f(y_s)[z] \mathbf{x}_{s,r}^{(1)} + f(y_r)[z] \mathbf{x}_{r,t}^{(1)} + \nabla f(y_s)[z] y \ltimes x_{s,r} + \nabla f(y_r)[z] y \ltimes x_{r,t} \\ &+ F(y_s)[z] \mathbf{x}_{s,r}^{(2)} + F(y_r)[z] \mathbf{x}_{r,t}^{(2)} + f(y_r) f(y_s)[z] \mathbf{x}_{s,r}^{(1)} \otimes \mathbf{x}_{r,t}^{(1)} + \sigma(s,r,t)[z]\end{aligned}$$

where $\sigma(s,r,t)[z]$ contains all the terms which are smaller than $C|z|\omega(s,t)^{3p}$ for some constant C. Since $\mathbf{x}_{s,t} = \mathbf{x}_{s,r} \otimes \mathbf{x}_{r,t}$ and then $\mathbf{x}^{(2)}_{s,t} = \mathbf{x}^{(2)}_{s,r} + \mathbf{x}^{(2)}_{r,t} + \mathbf{x}^{(1)}_{s,r} \otimes \mathbf{x}^{(1)}_{r,t}$ and using standard computations, this proves that $|\mathsf{A}_{s,t} - \mathsf{A}_{r,t}\mathsf{A}_{s,r}|C\omega(s,t)^{\theta}$ and then that A is an almost p-resolvent.

Let $\mathsf{B} = \mathsf{Id} + \mathbf{B}^{(1)} + \mathbf{B}^{(2)}$ be the p-rough resolvent associated to A through Theorem 1. It is easily checked that $\mathbf{B}^{(1)}_{s,t} = \mathcal{A}_t - \mathcal{A}_s$ for any $(s,t) \in \Delta_2(T)$ and that B is a p-left rough lift of $\mathcal{A}$. □

Proposition 12. *Under the above hypotheses on f, $\mathbf{x}$ and y, the three notions (linear RDE, in the sense of A.M. Davie and in the sense of Lyons) of solution to* (47) *are equivalent.*

Proof. For $z_t = \mathsf{A}_{0,t}a$, this proves that $z \in \mathcal{R}^p(\mathrm{W})$ and that it satisfies (48). It is then a solution in the sense of A.M. Davie, and then in the sense of T. Lyons (See [30] for the relationship between solutions in the sense of T. Lyons and int he sense of A.M. Davie).

The latter statement means that one may construct a family $\{z \ltimes x_{s,t}\}_{(s,t)\in\Delta_2(T)}$ with the same properties as $y \ltimes z$ such that $\mathbf{u} = (\mathbf{x}, y, z, y\ltimes x, z\ltimes x)$ is solution to the fixed point equation $\mathbf{u}_t = \mathbf{u}_0 + \int_0^t \underline{f}(u_s)\,\mathrm{d}\mathbf{u}_s$ where $\underline{f}$ is the differential form $\underline{f}(x,y,z) = \mathrm{d}x + \mathrm{d}y + f(y)z\,\mathrm{d}x$. Then, z is a path in $\mathcal{R}^p(\mathrm{W})$ satisfying (48), which is then a solution in a sense of A.M. Davie of (47). Then it follows from Corollary 3 that $z_t = \mathsf{A}_{0,t}z_0$. □

Let us consider now bounded function $g : \mathrm{V} \to L(\mathrm{U}, \mathrm{W})$ with a bounded derivative which is γ-Hölder continuous and set

$$b_t := \int_0^t g(y_s)\,\mathrm{d}\mathbf{x}_s.$$

This integral is well defined as the rough integral associated to the functional $g(y_s)\mathbf{x}^{(1)}_{s,t} + \nabla g(y_s) y \ltimes x_{s,t}$, and $b \in \mathcal{R}^p(\mathrm{W})$ in the sense that

$$|b_{s,t} - g(y_s)\mathbf{x}^{(1)}_{s,t} - \nabla g(y_s) y \ltimes x_{s,t}| \leq C\omega(s,t)^{\theta}$$

for some $\theta > 1$.

Our second aim is to consider the solution to the differential equation

$$z_t = a + \int_0^t f(y_s) z_s\,\mathrm{d}\mathbf{x}_s + \int_0^t g(y_s)\,\mathrm{d}\mathbf{x}_s. \tag{49}$$

Proposition 13. *Under the above hypotheses on f, $\mathbf{x}$ and y, there exists a unique solution to* (49) *and for any $t \in [0,T]$, $a \mapsto z_t$ defines an homeomorphism from* W *to* W. *Besides,*

$$\|z\|_p \leq C(|a| + \|g\|_\infty + \|\nabla g\|_\infty + H_\gamma(\nabla g)), \tag{50}$$

where C depends only on $\|f\|_\infty$ $\|\nabla f\|_\infty$, $H_\gamma(\nabla f)$, $\mathbf{x}$, $\omega(0,T)$, p and γ, and $H_\gamma(\nabla g)$ is the γ-Hölder norm of ∇g.

Remark 15. Similarly as above, $\int \nabla f(y_u) z_u \,\mathrm{d}\mathbf{x}_u$ may be understood as a rough integral.

Proof. Let us set

$$\mu_{s,t} := 1 + \int_s^t f(y_r)\,\mathrm{d}\mathbf{x}_r + \int_s^t g(y_r)\,\mathrm{d}\mathbf{x}_r + g(y_s)\otimes f(y_s)\mathbf{x}^{(2)}_{s,t}$$

Keeping only the terms in $\mathrm{X} := 1 \oplus L(\mathrm{W},\mathrm{W}) \oplus \mathrm{W} \oplus (\mathrm{W}\otimes L(\mathrm{W},\mathrm{W})))$, we have that

$$\begin{aligned}\mu_{s,t} - \mu_{s,r}\otimes\mu_{r,t} = {} & (g(y_s)\otimes f(y_s) - g(y_r)\otimes f(y_r))\mathbf{x}^{(2)}_{r,t} \\ & + \left(g(y_s)\mathbf{x}^{(1)}_{s,r} - \int_s^r g(y_u)d\mathbf{x}_u\right)\otimes f(y_s)\mathbf{x}^{(1)}_{r,t} \\ & + \int_s^r g(y_u)d\mathbf{x}_u \otimes \left(f(y_s)\mathbf{x}^{(1)}_{r,t} - \int_r^t f(y_u)d\mathbf{x}_u\right).\end{aligned}$$

Using the regularity of f and g and the properties of the rough integrals, it follows that $\{\mu_{s,t}\}_{(s,t)\in\Delta_2(T)}$ is an almost multiplicative functional to which is associated a family $\{\nu_{s,t}\}_{(s,t)\in\Delta_2(T)}$ with values in X satisfying $\nu_{s,t} = \nu_{s,r}\otimes\nu_{r,t}$ and so that

$$\nu^{(1)}_{s,t} = \int_s^t f(y_s)\,\mathrm{d}\mathbf{x}_s + \int_s^t g(y_s)\,\mathrm{d}\mathbf{x}_s \text{ and } \nu^{(2)}_{s,t} = \nu^{(2)}_{s,r} + \nu^{(2)}_{r,t} + b_{s,r}\otimes\mathbf{B}^{(1)}_{r,t}.$$

Let us now introduce the linear map from $(\mathrm{X},+)$ to $(\mathrm{W},+)$ defined by $\Phi(a) = \Phi(b) = \Phi(c) = 0$ and $\Phi(a\otimes b) := b[a]$ for $a\in\mathrm{W}$, $b\in L(\mathrm{W},\mathrm{W})$ and $c\in\mathbb{R}$.

Then $\mathbf{b}_{s,t} := \Phi(\nu_{s,t})$ is a left p-lift of b.

Using Proposition 9, this proves (50) and that the map $\Phi_{s,t}$ defined by

$$\Phi_{s,t}(a) := z_t \text{ with } z_t = a + \int_s^t f(y_u)z_r\,\mathrm{d}\mathbf{x}_u + \int_s^t g(y_s)\,\mathrm{d}\mathbf{x}_s,\ r\in[s,t] \tag{51}$$

defines a flow of homeomorphisms. □

Equations of type (51) are important since they appear when ones consider to differentiate parametrized solutions of

$$y_t = a + \int_0^t V(y_s,\lambda)\,\mathrm{d}\mathbf{x}_s + \int_0^t V(y_s,\lambda)\,\mathrm{d}h_s,\ h\in\mathcal{R}^q(\mathrm{V}) \text{ with } 1/p+1/q>1$$

with respect to a, λ and h. In this case, $f(\cdot) = \nabla V(\cdot,\lambda)$. Propositions 9 and 10, as well as Corollary 3, will then be used to study the regularity of the map $(a,\mathbf{x},h)\mapsto y$ in [15].

References

[1] S. Aida, *Notes on Proofs of Continuity Theorem in Rough Path Analysis* (2006). Unpublished note, Osaka University.

[2] A. Baker, *Matrix groups*, Springer Undergraduate Mathematics Series, Springer-Verlag London Ltd., London, 2002. An introduction to Lie group theory.

[3] F. Baudoin, *An Introduction to the Geometry of Stochastic Flows*, Imperial College Press, 2004.

[4] F. Baudoin and X. Zhang, *Taylor expansion for the solution of a stochastic differential equation driven by fractional Brownian motions*, Electronic Journal Probab. **17** (2012), no. 51, 1–21, DOI 10.1214/EJP.v17-2136.

[5] G. Ben Arous, *Flots et séries de Taylor stochastiques*, Probab. Theory Related Fields **81** (1989), no. 1, 29–77, DOI 10.1007/BF00343737.

[6] S. Blanes, F. Cases, J.A. Oteo, and J. Ros, *The Magnus expansion and some of its applications*, Physics Reports **470** (2009), 151-238.

[7] A. Bonfiglioli and R. Fulci, *Topics in Noncommutative Algebra. The Theorem of Campbell, Baker, Hausdorff and Dynkin*, Lecture Notes in Mathematics, vol. 2034, Springer-Verlag, 2012.

[8] M. Caruana and P. Friz, *Partial differential equations driven by rough paths*, J. Differential Equations **247** (2009), no. 1, 140–173, DOI 10.1016/j.jde.2009.01.026.

[9] M. Caruana, P. K. Friz, and Harald Oberhauser, *A (rough) pathwise approach to a class of non-linear stochastic partial differential equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire **28** (2011), no. 1, 27–46, DOI 10.1016/j.anihpc.2010.11.002.

[10] T. Cass, C. Litterer, and T. Lyons, *Integrability Estimates for Gaussian Rough Differential Equations* (2011), available at `arxiv:1104.1813`.

[11] F. Castell, *Asymptotic expansion of stochastic flows*, Probab. Theory Related Fields **96** (1993), no. 2, 225–239, DOI 10.1007/BF01192134.

[12] F. Castell and J. Gaines, *The ordinary differential equation approach to asymptotically efficient schemes for solution of stochastic differential equations*, Ann. Inst. H. Poincaré Probab. Statist. **32** (1996), no. 2, 231–250.

[13] K.-T. Chen, *Integration of paths, geometric invariants and a generalized Baker-Hausdorff formula*, Ann. of Math. (2) **65** (1957), 163–178.

[14] ———, *Formal differential equations*, Ann. of Math. (2) **73** (1961), 110–133.

[15] L. Coutin and A. Lejay, *Sensitivity of rough differential equations* (2012). Preprint.

[16] A.M. Davie, *Differential Equations Driven by Rough Signals: an Approach via Discrete Approximation*, Appl. Math. Res. Express. AMRX **2** (2007), Art. ID abm009, 40.

[17] A. Deya and S. Tindel, *Rough Volterra equations. I. The algebraic integration setting*, Stoch. Dyn. **9** (2009), no. 3, 437–477, DOI 10.1142/S0219493709002737.

[18] ———, *Rough Volterra equations 2: convolutional generalized integrals*, Stochastic Process. Appl. **121** (2011), no. 8, 1864–1899, DOI 10.1016/j.spa.2011.05.003.

[19] R. G. Douglas, *Banach algebra techniques in operator theory*, 2nd ed., Graduate Texts in Mathematics, vol. 179, Springer-Verlag, New York, 1998.

[20] F.J. Dyson, *The Radiation Theories of Tomonaga, Schwinger, and Feynman*, Phys. Rev. **75** (1949), 486-502.

[21] D. Feyel, A. de La Pradelle, and G. Mokobodzki, *A non-commutative sewing lemma*, Electron. Commun. Probab. **13** (2008), 24–34.

[22] M. Fliess and D. Normand-Cyrot, *Algèbres de Lie nilpotentes, formule de Baker-Campbell-Hausdorff et intégrales itérées de K. T. Chen*, Seminar on Probability, XVI, Lecture Notes in Math., vol. 920, Springer, Berlin, 1982, pp. 257–267.

[23] P. Friz and N. Victoir, *Multidimensional Stochastic Processes as Rough Paths. Theory and Applications*, Cambridge University Press, 2010.

[24] ______, *A note on the notion of geometric rough paths*, Probab. Theory Related Fields **136** (2006), no. 3, 395–416, DOI 10.1007/s00440-005-0487-7.

[25] M. Gubinelli, *Abstract integration, combinatorics of trees and differential equations*, Combinatorics and physics, Contemp. Math., vol. 539, Amer. Math. Soc., Providence, RI, 2011, pp. 135–151.

[26] Deya. A., M. Gubinelli, and S. Tindel, *Non-linear Rough Heat Equation*, Probab. Theory Related Fields **153** (2012), no. 1–2, 97–147, DOI 10.1007/s00440-011-0341-z.

[27] M. Gubinelli, A. Lejay, and S. Tindel, *Young integrals and SPDEs*, Potential Anal. **25** (2006), no. 4, 307–326, DOI 10.1007/s11118-006-9013-5.

[28] E. Hairer, C. Lubich, and G. Wanner, *Geometric numerical integration: Structure-preserving algorithms for ordinary differential equations*, Springer Series in Computational Mathematics, vol. 31, Springer, Heidelberg, 2010.

[29] B. C. Hall, *Lie groups, Lie algebras, and representations*, Graduate Texts in Mathematics, vol. 222, Springer-Verlag, New York, 2003. An elementary introduction.

[30] A. Lejay, *Global solutions to rough differential equations with unbounded vector fields*, Séminaire de Probabilités XLIV, Lecture Notes in Mathematics, vol. 2046, Springer-Verlag, 2012, pp. 215–246.

[31] ______, *On rough differential equations*, Electron. J. Probab. **14** (2009), no. 12, 341–364.

[32] ______, *Yet another introduction to rough paths*, Séminaire de probabilités XLII, Lecture Notes in Mathematics, vol. 1979, Springer-Verlag, 2009, pp. 1–101.

[33] A. Lejay and N. Victoir, *On (p,q)-rough paths*, J. Differential Equations **225** (2006), no. 1, 103–133, DOI 10.1016/j.jde.2006.01.018.

[34] A. Lejay, *An introduction to rough paths*, Séminaire de probabilités, XXXVII, Lecture Notes in Mathematics, vol. 1832, Springer-Verlag, 2003, pp. 1–59.

[35] G. Lord, S. J. A. Malham, and A. Wiese, *Efficient strong integrators for linear stochastic systems*, SIAM J. Numer. Anal. **46** (2008), no. 6, 2892–2919, DOI 10.1137/060656486.

[36] T. Lyons and W. Xu, *A uniform estimate for rough paths* (2011), available at `arXiv:1110.5278`.

[37] T. J. Lyons and N. Sidorova, *On the radius of convergence of the logarithmic signature*, Illinois J. Math. **50** (2006), no. 1-4, 763–790.

[38] T.J. Lyons, *Differential equations driven by rough signals*, Rev. Mat. Iberoamericana **14** (1998), no. 2, 215–310.

[39] T. Lyons and Z. Qian, *System Control and Rough Paths*, Oxford Mathematical Monographs, Oxford University Press, 2002.

[40] C. Moan and J. Niesen, *Convergence of the Magnus series*, Found. Comput. Math. **8** (2008), no. 3, 291–301, DOI 10.1007/s10208-007-9010-0.

[41] R. Ree, *Lie elements and an algebra associated with shuffles*, Ann. of Math. (2) **68** (1958), 210–220.

[42] R. S. Strichartz, *The Campbell-Baker-Hausdorff-Dynkin formula and solutions of differential equations*, J. Funct. Anal. **72** (1987), no. 2, 320–345, DOI 10.1016/0022-1236(87)90091-7.

[43] Y. Yamato, *Stochastic differential equations and nilpotent Lie algebras*, Z. Wahrsch. Verw. Gebiete **47** (1979), no. 2, 213–229, DOI 10.1007/BF00535284.

[44] L.C. Young, *An inequality of the Hölder type, connected with Stieltjes integration*, Acta Math. **67** (1936), 251–282.